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RESEARCH REPORT

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Factorized filtering with simultaneous data and time updating

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Chapter 1

Introduction

The paper presents the modified algorithm of factorized Kalman filtering. A modification of the algorithm consists in simultaneous fulfillment of the data and time updating of the posterior factorized state estimate for system, whose state and output are described by the joint pdf. Such the modification enables to use multi-output normal observation model.

The state of the art of the factorized filtering can be summarized by description of several research works in the considered domain. The paper [1] proposed the recursive algorithm of the entry-wise organized filtering under Bayesian methodology [2], restricted by the reduced form of the state-space model, concluded in the triangular transition matrix. The papers [3, 4] removed this restriction and proposed the solution of factorized Bayesian prediction and filtering, based on applying the chain rule to the single output state-space model. The work [5] offered the version of factorized Kalman filtering with Gaussian models, which was based on the $L'DL$ decomposition of the covariance matrices. The paper [6] expanded the line with $L'DL$-factorized covariance matrices and demonstrated the application of the solution to the traffic system state-space model. The work [7] proposed a version of the full factorization of the observation model. Moreover, a methodology of specification of prior distributions of the individual state factors, based on approaches from [8, 9] has been proposed in [7]. The present work continues research in the field of factorized filtering. It modifies the algorithm, as it was mentioned above, and improved the technique of exploitation of the full factorized observation model in filtering.

The paper outline includes the following parts. Sections 2.1-2.2 describes the models used and provides derivation of the simultaneous data and time updating and its factorized version. Section 2.3 proposes the factorized Kalman filter with the simultaneous data and time updating, providing the key moments of the filtering algorithm. Section 2.4 presents the example of application of the proposed algorithm to Gaussian state-space model with three state and output factors.

Chapter 2

Bayesian filtering

2.1 Models

The system is described by the joint probability density function (pdf)

$$f\left(x_t, y_t \middle| x_{t-1}, u_t\right), \tag{2.1}$$

where x_t is the system state, y_t is the system output and u_t is the system input. This joint pdf can be, according to the chain rule [10], decomposed into the following factorized form

$$\prod_{i=1}^{\mathring{x}} f\left(x_{i;t} \middle| x_{i+1:\mathring{x};t}, u_t, x_{t-1}, y_t\right) \prod_{j=1}^{\mathring{y}} f\left(y_{j;t} \middle| y_{j+1:\mathring{y};t}, u_t, x_{t-1}\right), \tag{2.2}$$

where $\mathring{x}$ and $\mathring{y}$ are the numbers of entries of respective state and output vectors and notation such as $x_{i+1:\mathring{x};t}$ denotes a sequence $\{x_{i+1;t}, x_{i+2;t}, \ldots, x_{\mathring{x};t}\}$.

When speaking about application of filtering to Gaussian state-space models, the system is assumed to be described by

$$\begin{aligned} &\textit{state evolution model} \qquad & x_t &= Ax_{t-1} + Bu_t + \omega_t, & (2.3)\\ &\textit{observation model} \qquad & y_t &= Cx_{t-1} + Hu_t + v_t, & (2.4) \end{aligned}$$

where ω_t is a process (Gaussian) noise with zero mean and covariance (inverse) matrix R_w; v_t is a measurement (Gaussian) noise with zero mean and covariance (already inverse) matrix R_v; A, B, C and H are known matrices of appropriate dimensions.

2.2 Factorized Bayesian filtering

Manupilation with the state-space model in the form of joint pdf allows to rewrite the Bayesian filtering so that the data and time updating are fulfilled simultaneously. It is obtained from Bayes rule [10]

$$\begin{aligned} f\left(y_t, x_t \middle| u_t, d^{1:t-1}\right) &= f\left(y_t \middle| u_t, d^{1:t-1}, x_t\right) f\left(x_t \middle| u_t, d^{1:t-1}\right), & (2.5)\\ &= f\left(x_t \middle| d^{1:t}\right) f\left(y_t \middle| u_t, d^{1:t-1}\right), & (2.6) \end{aligned}$$

which gives the data updating as

$$f\left(x_t \middle| d^{1:t}\right) = \frac{f\left(y_t, x_t \middle| u_t, d^{1:t-1}\right)}{f\left(y_t \middle| u_t, d^{1:t-1}\right)}, \tag{2.7}$$

with Bayesian predictor [10] as a denominator.

Then, with the help of operation of marginalization and according to the used model, one can obtain the simultaneous data & time updating

$$
\begin{aligned}
f\left(x_t \middle| d^{1:t}\right) &= \frac{f\left(y_t, x_t \middle| u_t, d^{1:t-1}\right)}{f\left(y_t \middle| u_t, d^{1:t-1}\right)}, && (2.8)\\
&= \frac{\int f\left(x_t, y_t, x_{t-1} \middle| u_t, d^{1:t-1}\right)\, dx_{t-1}}{f\left(y_t \middle| u_t, d^{1:t-1}\right)}, && (2.9)\\
&= \frac{\int f\left(x_t, y_t \middle| u_t, x_{t-1}\right) f\left(x_{t-1} \middle| d^{1:t-1}\right)\, dx_{t-1}}{f\left(y_t \middle| u_t, d^{1:t-1}\right)}, && (2.10)\\
&\propto \int f\left(x_t, y_t \middle| u_t, x_{t-1}\right) f\left(x_{t-1} \middle| d^{1:t-1}\right)\, dx_{t-1}, && (2.11)
\end{aligned}
$$

To use the simultaneous data & time updating (2.11) in the factorized form, it is necessary to replace the joint pdf $f\left(x_t, y_t \middle| u_t, x_{t-1}\right)$ by the product of factors from (2.2) and to factorize the initial state in the following way.

$$
f\left(x_{t-1} \middle| d^{1:t-1}\right) = \prod_{i=1}^{\mathring{x}} f\left(x_{i;t-1} \middle| x_{i+1:\mathring{x};t-1}, d^{1:t-1}\right), \qquad (2.12)
$$

The simultaneous data & time updating, consequently, is evolved as

$$
\begin{aligned}
& f\left(x_t \middle| d^{1:t}\right) && (2.13)\\
\propto & \int \prod_{i=1}^{\mathring{x}} f\left(x_{i;t} \middle| x_{i+1:\mathring{x};t}, u_t, x_{t-1}, y_t\right) \prod_{j=1}^{\mathring{y}} f\left(y_{j;t} \middle| y_{j+1:\mathring{y};t}, u_t, x_{t-1}\right) \prod_{i=1}^{\mathring{x}} f\left(x_{i;t-1} \middle| x_{i+1:\mathring{x};t-1}, d^{1:t-1}\right)\, dx_{t-1}.
\end{aligned}
$$

2.3 Algorithm of factorized Kalman filtering

The application of factorized Bayesian filtering to Gaussian state-space model leads to the factorized version of Kalman filter [11]. The key moments of this filter algorithm are as follows.

2.3.1 Initial state factorization

The covariance (inverse) matrix P of the initial state $x_{t-1} \sim \mathcal{N}_{x_{t-1}}(\mu, P)$ is factorized through the following decomposition

$$
P = LDL', \qquad (2.14)
$$

where L is a lower triangular matrix with unit diagonal and D is a diagonal matrix. Vector $\mu = [\mu_1 \quad \dots \quad \mu_{\mathring{x}}]'$ contains the (known) initial mean values of the state vector entries. It means, that the factorized form of the initial state distribution (2.12) $f\left(x_{t-1} \middle| d^{1:t-1}\right) = \prod_{i=1}^{\mathring{x}} f\left(x_{i;t-1} \middle| x_{i+1:\mathring{x};t-1}, d^{1:t-1}\right)$ for Gaussian models takes the form

$$
\mathcal{N}_{x_{i;t-1}}\left(\hat{\mu}_{i;t-1} - \sum_{k=i+1}^{\mathring{x}} l_{ki} x_{k;t-1}, \hat{p}_{i;t-1}\right), \qquad (2.15)
$$

where l_{ki} are the elements of matrix L; $\hat{p}_{i;t-1} = \frac{1}{d_{ii}}$, where d_{ii} are the elements of matrix D; and $\hat{\mu}_{i;t-1} = \mu_i + \sum_{k=i+1}^{\mathring{x}} l_{ki}\mu_k$.

2.3.2 Factorization of state evolution model

Normal state evolution model (2.3) is factorized in the similar way. The process covariance (inverse) matrix is decomposed so that

$$
R_w = GD^G G', \qquad (2.16)
$$

where G is a lower triangular matrix with unit diagonal and D^G is a diagonal matrix. The individual state entries have the following form of normal distribution.

$$
\mathcal{N}_{x_{i;t}}(z_i - \sum_{k=i+1}^{\mathring{x}} g_{ki} x_{k;t} + \sum_{l=1}^{\mathring{x}} \xi_{il} x_{l;t-1}, p_{ii}^G), \qquad (2.17)
$$

where

$$z_i = BU_i + \sum_{k=i+1}^{\mathring{x}} BU_k g_{ki}, \tag{2.18}$$

$$BU_i = [b_{i1} \quad \dots \quad b_{is}]u_t, s = \{1, \dots, \mathring{u}\}, \tag{2.19}$$

$$\xi_{il} = a_{il} + \sum_{k=i+1}^{\mathring{x}} a_{kl} g_{ki}, \tag{2.20}$$

$$p_{ii}^G = \frac{1}{d_{ii}^G}, \tag{2.21}$$

where a_{il} and b_{is} are elements of matrices A and B in (2.3) respectively, and g_{ki} and d_{ii}^G are elements of matrices G and D^G respectively from (2.16).

2.3.3 Factorization of observation model

The normal observation model (2.4) is factorized in the following way. The measurement covariance (inverse) matrix is decomposed as

$$R_v = L^{Rv} D^{Rv} L^{Rv\prime}, \tag{2.22}$$

where, again, L^{Rv} is a lower triangular matrix with unit diagonal and D^{Rv} is a diagonal matrix. The individual output entries have the following form of normal distribution.

$$\mathcal{N}_{y_{j;t}}(\rho_j - \sum_{k=j+1}^{\mathring{y}} l_{kj}^{Rv} y_{k;t} + \sum_{i=1}^{\mathring{x}} \alpha_{ji} x_{i;t-1}, p_{jj}^{Rv}), \tag{2.23}$$

where

$$\rho_j = HU_j + \sum_{k=j+1}^{\mathring{y}} HU_k l_{kj}^{Rv}, \tag{2.24}$$

$$HU_j = [h_{j1} \quad \dots \quad h_{js}]u_t, s = \{1, \dots, \mathring{u}\}, \tag{2.25}$$

$$\alpha_{ji} = c_{ji} + \sum_{k=j+1}^{\mathring{y}} c_{ki} l_{kj}^{Rv}, \tag{2.26}$$

$$p_{jj}^{Rv} = \frac{1}{d_{jj}^{Rv}}, \tag{2.27}$$

$$\tag{2.28}$$

where c_{ji} and h_{js} are elements of matrices C and H in (2.4) respectively.

2.3.4 Factorized simultaneous data & time updating

The algorithm of simultaneous data & time updating (2.13) is proposed as follows. The relation to be integrated in (2.13) is the product of Gaussian pdfs and can be presented in the form

$$\int \prod_{k=1}^{\mathring{\beta}^{(m)}} \exp\left\{-\frac{(\beta_k^{(m)} - \gamma_k^{(m)} x_{m;t-1})^2}{2r_k^{(m)}}\right\} dx_{m;t-1} \propto \exp\left\{-\frac{\lambda^{(m)}}{2}\right\}, \tag{2.29}$$

$$\text{with } \lambda^{(m)} = \beta^{(m)\prime}\left(\omega^{(m)} - \frac{\omega^{(m)}\gamma^{(m)}\gamma^{(m)\prime}\omega^{(m)}}{\gamma^{(m)\prime}\omega^{(m)}\gamma^{(m)}}\right)\beta^{(m)} = \sum_{k=1}^{\mathring{\beta}^{(m)}} \frac{\left(\beta_k^{(m)} + \sum_{l=k+1}^{\mathring{\beta}^{(m)}} \beta_l^{(m)} U_{lk}^{(m)}\right)^2}{\frac{1}{D_{kk}^{(m)}}}, \tag{2.30}$$

where the upper right index (m), $m = \{1, \dots, \mathring{x}\}$ denotes the index of the factor $x_{m;t-1}$, being currently integrated out. Vector $\beta^{(m)}$ is defined as $[\beta_1^{(m)}, \dots, \beta_{\mathring{\beta}^{(m)}}^{(m)}]'$, where $\beta_k^{(m)}$, $k = \{1, \dots, \mathring{\beta}^{(m)}\}$ presents the sum of all members but the being integrated one (i.e. $x_{m;t-1}$) in the quadratic form of the k-th Gaussian pdf in (2.13). The capacity $\mathring{\beta}^{(m)}$ is composed from $\mathring{x} + \mathring{y} + m$. The additional unit(s) m appear due to Gaussian pdfs of factor(s) $x_{m;t-1}$, being currently integrated out or having been already integrated ones. Vector $\gamma^{(m)}$ is defined as $[\gamma_1^{(m)},\ldots,\gamma_{\mathring{\beta}(m)}^{(m)}]'$, where $\gamma_k^{(m)}$ is a coefficient, corresponding to the factor $x_{m;t-1}$, being integrated out, in the k-th Gaussian pdf in (2.13). For coefficients $\gamma_k^{(m)}$ the following recursions over $m=\{1,\ldots,\mathring{x}\}$ hold.

$$\begin{aligned}\gamma^{(m)} &= [\gamma_1^{(m)},\ldots,\gamma_{\mathring{\beta}(m)}^{(m)}]', && (2.31)\\ &= [\xi_{1m}^{(m-1)},\ldots,\xi_{\mathring{x}m}^{(m-1)},\alpha_{1m}^{(m-1)},\ldots,\alpha_{\mathring{y}m}^{(m-1)},l_{m1},\ldots,l_{m;\mathring{x}-1},1]', && (2.32)\end{aligned}$$

where

$$\begin{aligned}\xi_{il}^{(m)} &= \xi_{il}^{(m-1)}+\sum_{k=i+1}^{\mathring{x}}\xi_{kl}^{(m-1)}U_{ki}^{(m)}+\sum_{j=1}^{\mathring{y}}\alpha_{jl}^{(m-1)}U_{\mathring{x}+j;i}^{(m)}+\sum_{k=1}^{\mathring{x}-1}l_{lk}U_{\mathring{x}+\mathring{y}+k;i}^{(m)}, && (2.33)\\ \alpha_{jl}^{(m)} &= \alpha_{jl}^{(m-1)}+\sum_{k=j+1}^{\mathring{y}}\alpha_{kl}^{(m-1)}U_{\mathring{x}+k;\mathring{x}+j}^{(m)}+\sum_{k=1}^{\mathring{x}-1}l_{lk}U_{\mathring{x}+\mathring{y}+k;j}^{(m)}. && (2.34)\end{aligned}$$

Variances $r_k^{(m)}$ correspond to the k-th Gaussian pdf in (2.13) and compose the diagonal matrix $\omega^{(m)}\equiv\operatorname{diag}\left[r_1^{(m)^{-1}},\ldots,r_{\mathring{\beta}(m)}^{(m)\ -1}\right]$. The remainder $\lambda^{(m)}$ in (2.29) is obtained with the help of completion of squares for $x_{m;t-1}$ and integration of Gaussian pdfs, that is proved by direct calculation. The result in (2.30) is obtained via decomposition

$$\left(\omega^{(m)}-\frac{\omega^{(m)}\gamma^{(m)}\gamma^{(m)'}\omega^{(m)}}{\gamma^{(m)'}\omega^{(m)}\gamma^{(m)}}\right)=U^{(m)'}D^{(m)}U^{(m)}, \qquad (2.35)$$

where $U^{(m)}$ is upper triangular matrix with unit diagonal and $D^{(m)}$ is diagonal matrix.

After sequential integration over factors $x_{m;t-1}$, the joint pdf (2.2) preserves its form without $x_{m;t-1}$, and Gaussian state factors $x_{i;t}$, $i=\{1,\ldots,\mathring{x}\}$ have Gaussian distributions as follows.

$$\mathcal{N}_{x_{i;t}}\left(\mu_{i;t}^{(m)}-\sum_{k=i+1}^{\mathring{x}}g_{ki}^{(m)}x_{k;t}-\sum_{j=1}^{\mathring{y}}\eta_{ji}^{(m)}y_{j;t},p_{i;t}\right), \qquad (2.36)$$

where $\mu_{i;t}^{(m)}$ and coefficients $g_{ki}^{(m)}$ and $\eta_{ji}^{(m)}$ are state-independent. The following recursions over $m=\{1,\ldots,\mathring{x}\}$ hold

$$\begin{aligned}\mu_{i;t}^{(m)} &= \mu_{i;t}^{(m-1)}+\sum_{k=i+1}^{\mathring{x}}\mu_{k;t}^{(m-1)}U_{ki}^{(m)}+\sum_{j=1}^{\mathring{y}}\rho_j^{(m-1)}U_{\mathring{x}+j;i}^{(m)}-\sum_{k=1}^{m}\hat{\mu}_{k;t-1}U_{\mathring{x}+\mathring{y}+k;i}^{(m)}, && (2.37)\\ \rho_j^{(m)} &= \rho_j^{(m-1)}+\sum_{k=j+1}^{\mathring{y}}\rho_k^{(m-1)}U_{\mathring{x}+k;\mathring{x}+j}^{(m)}-\sum_{k=1}^{m}\hat{\mu}_{k;t-1}U_{\mathring{x}+\mathring{y}+k;\mathring{x}+j}^{(m)}, && (2.38)\\ g_{ki}^{(m)} &= g_{ki}^{(m-1)}+\sum_{l=i+1}^{k-1}g_{kl}^{(m-1)}U_{li}^{(m)}+U_{ki}^{(m)}, && (2.39)\\ \eta_{ji}^{(m)} &= \eta_{ji}^{(m-1)}+\sum_{k=i+1}^{\mathring{x}}\eta_{jk}^{(m-1)}U_{ki}^{(m)}+\sum_{k=i+1}^{\mathring{x}}l_{jk}^{Rv\,(m-1)}U_{\mathring{x}+k;i}^{(m)}+U_{\mathring{x}+j;i}^{(m)}, && (2.40)\\ l_{ji}^{Rv\,(m)} &= l_{ji}^{Rv\,(m-1)}+\sum_{l=i+1}^{j-1}l_{jl}^{Rv\,(m-1)}U_{\mathring{x}+l;\mathring{x}+i}^{(m)}+U_{\mathring{x}+j;\mathring{x}+i}^{(m)}, && (2.41)\\ p_{i;t} &= \frac{1}{D_{ii}^{(m)}}. && (2.42)\end{aligned}$$

The recursions start with m=1, for which $\mu_{i;t}^{(m-1)}=z_i$ from (2.18), $\rho_j^{(m-1)}=\rho_j$ from (2.24), $g_{ki}^{(m-1)}=g_{ki}$ according to (2.17), and $l_{ji}^{Rv\,(m-1)}=l_{ji}^{Rv}$ according to (2.23). Coefficients $\eta_{ji}^{(m)}$ are evolved precisely according to (2.40).

Gaussian output factors $y_{j;t}$, $j=\{1,\ldots,\mathring{y}\}$, similarly, obtain Gaussian distributions as follows.

$$\mathcal{N}_{y_{j;t}}\left(\rho_j^{(m)}-\sum_{k=j+1}^{\mathring{y}}l_{kj}^{Rv\,(m)}y_{k;t},\frac{1}{D_{\mathring{x}+j;\mathring{x}+j}^{(m)}}\right), \qquad (2.43)$$

with $\rho_j^{(m)}$ defined in (2.38) and

$$l_{kj}^{Rv(m)} = l_{kj}^{Rv(m-1)} + \sum_{l=i+1}^{k-1} l_{kl}^{Rv(m-1)} U_{\mathring{x}+l;\mathring{x}+i}^{(m)} + U_{\mathring{x}+k;\mathring{x}+i}^{(m)}. \tag{2.44}$$

The obtained results are proved by direct calculation.

The example of calculation of simultaneous data & time updating for three-dimensional state is given below.

2.4 Example of filtering for three-dimensional state and output

For the presented example the dimension of the state $\mathring{x} = 3$, dimension of the output $\mathring{y} = 3$, and dimension of the input $\mathring{u} = 2$.

The simultaneous data & time updating (2.13) for the considered example takes the form

$$\begin{aligned} & \int\int\int f\left(x_{1;t} | x_{2:3;t}, u_t, x_{1:3;t-1}, y_{1:3;t}\right) f\left(x_{2;t} | x_{3;t}, u_t, x_{1:3;t-1}, y_{1:3;t}\right) f\left(x_{3;t} | u_t, x_{1:3;t-1}, y_{1:3;t}\right), \\ \times \quad & f\left(y_{1;t} | y_{2:3;t}, u_t, x_{1:3;t-1}\right) f\left(y_{2;t} | y_{3;t}, u_t, x_{1:3;t-1}\right) f\left(y_{3;t} | u_t, x_{1:3;t-1}\right), \\ \times \quad & f\left(x_{1;t-1} \middle| x_{2:3;t-1}, d^{1:t-1}\right) f\left(x_{2;t-1} \middle| x_{3;t-1}, d^{1:t-1}\right) f\left(x_{3;t-1} \middle| d^{1:t-1}\right) dx_{1;t-1}\, dx_{2;t-1}\, dx_{3;t-1}. \end{aligned} \tag{2.45}$$

2.4.1 Initial state factorization

According to (2.14), the initial state distribution with the known mean vector μ and known inverse covariance matrix P is factorized in the following way.

$$\begin{aligned} & \begin{bmatrix} x_{1;t-1}-\mu_1 \\ x_{2;t-1}-\mu_2 \\ x_{3;t-1}-\mu_3 \end{bmatrix}' \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} x_{1;t-1}-\mu_1 \\ x_{2;t-1}-\mu_2 \\ x_{3;t-1}-\mu_3 \end{bmatrix}, \\ = & \begin{bmatrix} x_{1;t-1}-\mu_1 \\ x_{2;t-1}-\mu_2 \\ x_{3;t-1}-\mu_3 \end{bmatrix}' \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}' \begin{bmatrix} x_{1;t-1}-\mu_1 \\ x_{2;t-1}-\mu_2 \\ x_{3;t-1}-\mu_3 \end{bmatrix}, \\ = & \begin{bmatrix} (x_{1;t-1}-\mu_1)+(x_{2;t-1}-\mu_2)l_{21}+(x_{3;t-1}-\mu_3)l_{31} \\ (x_{2;t-1}-\mu_2)+(x_{3;t-1}-\mu_3)l_{32} \\ (x_{3;t-1}-\mu_3) \end{bmatrix}' \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}, \\ \times & \begin{bmatrix} (x_{1;t-1}-\mu_1)+(x_{2;t-1}-\mu_2)l_{21}+(x_{3;t-1}-\mu_3)l_{31} \\ (x_{2;t-1}-\mu_2)+(x_{3;t-1}-\mu_3)l_{32} \\ (x_{3;t-1}-\mu_3) \end{bmatrix}. \end{aligned} \tag{2.46}$$

It means, that according to (2.15), the normal distributions of the initial state factors take the form

$$\mathcal{N}_{x_{1;t-1}}\left(\underbrace{\mu_1 + l_{21}\mu_2 + l_{31}\mu_3}_{\hat{\mu}_{1;t-1}} - l_{21}x_{2;t-1} - l_{31}x_{3;t-1}, \hat{p}_{1;t-1}\right), \text{with } \hat{p}_{1;t-1} = \frac{1}{d_{11}}, \tag{2.47}$$

$$\mathcal{N}_{x_{2;t-1}}\left(\underbrace{\mu_2 + l_{32}\mu_3}_{\hat{\mu}_{2;t-1}} - l_{32}x_{3;t-1}, \hat{p}_{2;t-1}\right), \text{with } \hat{p}_{2;t-1} = \frac{1}{d_{22}}, \tag{2.48}$$

$$\mathcal{N}_{x_{3;t-1}}\left(\underbrace{\mu_3}_{\hat{\mu}_{3;t-1}}, \hat{p}_{3;t-1}\right), \text{with } \hat{p}_{3;t-1} = \frac{1}{d_{33}}. \tag{2.49}$$

2.4.2 Factorization of state evolution model

Similarly, factorization of state evolution model (2.3) according to (2.16) and (2.17) enables to describe the state factors in the following way.

$$\mathcal{N}_{x_{1;t}}(z_1 - g_{21}x_{2;t} - g_{31}x_{3;t} + \xi_{11}x_{1;t-1} + \xi_{12}x_{2;t-1} + \xi_{13}x_{3;t-1}, p_{11}^G), \tag{2.50}$$

where

$$\begin{aligned} z_1 &= BU_1 + BU_2 g_{21} + BU_3 g_{31},\ BU_1 = [b_{11} \;\; b_{12}]u_t,\ BU_2 = [b_{21} \;\; b_{22}]u_t,\ BU_3 = [b_{31} \;\; b_{32}]u_t, \\ \xi_{11} &= a_{11} + a_{21}g_{21} + a_{31}g_{31},\ \ \xi_{12} = a_{12} + a_{22}g_{21} + a_{32}g_{31}, \\ \xi_{13} &= a_{13} + a_{23}g_{21} + a_{33}g_{31},\ \ p^G_{11} = \frac{1}{d^G_{11}}, \end{aligned}$$

$$\mathcal{N}_{x_{2;t}}(z_2 - g_{32}x_{3;t} + \xi_{21}x_{1;t-1} + \xi_{22}x_{2;t-1} + \xi_{23}x_{3;t-1}, p^G_{22}), \tag{2.51}$$

where

$$\begin{aligned} z_2 &= BU_2 + BU_3 g_{32},\ \ \xi_{21} = a_{21} + a_{31}g_{32}, \\ \xi_{22} &= a_{22} + a_{32}g_{32},\ \ \xi_{23} = a_{23} + a_{33}g_{32},\ \ p^G_{22} = \frac{1}{d^G_{22}}, \end{aligned}$$

$$\mathcal{N}_{x_{3;t}}(z_3 + \xi_{31}x_{1;t-1} + \xi_{32}x_{2;t-1} + \xi_{33}x_{3;t-1}, p^G_{33}), \tag{2.52}$$

where

$$z_3 = BU_3,\ \ \xi_{31} = a_{31},\ \ \xi_{32} = a_{32},\ \ \xi_{33} = a_{33},\ \ p^G_{33} = \frac{1}{d^G_{33}}.$$

2.4.3 Factorization of observation model

In the similar way, factorization of the observation model (2.4) is fulfilled via decomposition (2.22).

$$\begin{aligned} &\begin{bmatrix} (y_{1;t} - \hat{y}_{1;t} - HU_1) + (y_{2;t} - \hat{y}_{2;t} - HU_2)l^{Rv}_{21} + (y_{3;t} - \hat{y}_{3;t} - HU_3)l^{Rv}_{31} \\ (y_{2;t} - \hat{y}_{2;t} - HU_2) + (y_{3;t} - \hat{y}_{3;t} - HU_3)l^{Rv}_{32} \\ (y_{3;t} - \hat{y}_{3;t} - HU_3) \end{bmatrix}' \begin{bmatrix} d^{Rv}_{11} & 0 & 0 \\ 0 & d^{Rv}_{22} & 0 \\ 0 & 0 & d^{Rv}_{33} \end{bmatrix}, \\ \times &\begin{bmatrix} (y_{1;t} - \hat{y}_{1;t} - HU_1) + (y_{2;t} - \hat{y}_{2;t} - HU_2)l^{Rv}_{21} + (y_{3;t} - \hat{y}_{3;t} - HU_3)l^{Rv}_{31} \\ (y_{2;t} - \hat{y}_{2;t} - HU_2) + (y_{3;t} - \hat{y}_{3;t} - HU_3)l^{Rv}_{32} \\ (y_{3;t} - \hat{y}_{3;t} - HU_3) \end{bmatrix}, \end{aligned} \tag{2.53}$$

where

$$\hat{y}_{1;t} = c_{11}x_{1;t-1} + c_{12}x_{2;t-1} + c_{13}x_{3;t-1},\ \ HU_1 = [h_{11} \;\; h_{12}]u_t, \tag{2.54}$$

$$\hat{y}_{2;t} = c_{21}x_{1;t-1} + c_{22}x_{2;t-1} + c_{23}x_{3;t-1},\ \ HU_2 = [h_{21} \;\; h_{22}]u_t, \tag{2.55}$$

$$\hat{y}_{3;t} = c_{31}x_{1;t-1} + c_{32}x_{2;t-1} + c_{33}x_{3;t-1},\ \ HU_3 = [h_{31} \;\; h_{32}]u_t. \tag{2.56}$$

The output factors, consequently, are described as follows.

$$\mathcal{N}_{y_{1;t}}(\rho_1 - l^{Rv}_{21}y_{2;t} - l^{Rv}_{31}y_{3;t} + \alpha_{11}x_{1;t-1} + \alpha_{12}x_{2;t-1} + \alpha_{13}x_{3;t-1}, p^{Rv}_{11}), \tag{2.57}$$

with

$$\begin{aligned} \rho_1 &= HU_1 + HU_2 l^{Rv}_{21} + HU_3 l^{Rv}_{31},\ \ \alpha_{11} = c_{11} + c_{21}l^{Rv}_{21} + c_{31}l^{Rv}_{31}, \\ \alpha_{12} &= c_{12} + c_{22}l^{Rv}_{21} + c_{32}l^{Rv}_{31},\ \ \alpha_{13} = c_{13} + c_{23}l^{Rv}_{21} + c_{33}l^{Rv}_{31},\ \ p^{Rv}_{11} = \frac{1}{d^{Rv}_{11}}, \end{aligned}$$

$$\mathcal{N}_{y_{2;t}}(\rho_2 - l^{Rv}_{32}y_{3;t} + \alpha_{21}x_{1;t-1} + \alpha_{22}x_{2;t-1} + \alpha_{23}x_{3;t-1}, p^{Rv}_{22}), \tag{2.58}$$

with

$$\rho_2 = HU_2 + HU_3 l^{Rv}_{32},\ \ \alpha_{21} = c_{21} + c_{31}l^{Rv}_{32},\ \ \alpha_{22} = c_{22} + c_{32}l^{Rv}_{32},\ \ \alpha_{23} = c_{23} + c_{33}l^{Rv}_{32},\ \ p^{Rv}_{22} = \frac{1}{d^{Rv}_{22}},$$

$$\mathcal{N}_{y_{3;t}}(\rho_3 + \alpha_{31}x_{1;t-1} + \alpha_{32}x_{2;t-1} + \alpha_{33}x_{3;t-1}, p^{Rv}_{33}), \tag{2.59}$$

with

$$\rho_3 = HU_3,\ \ \alpha_{31} = c_{31},\ \ \alpha_{32} = c_{32},\ \ \alpha_{33} = c_{33},\ \ p^{Rv}_{33} = \frac{1}{d^{Rv}_{33}}.$$

2.4.4 Integration over the first state factor

Integration of the product of Gaussian pdfs in (2.45) should be firstly done over the state factor $x_{1;t-1}$. According to (2.29), the relation to be integrated is the following one.

$$\int \exp\left\{-\frac{\left(\overbrace{x_{1;t}-z_1+g_{21}x_{2;t}+g_{31}x_{3;t}-\xi_{12}x_{2;t-1}-\xi_{13}x_{3;t-1}}^{\beta_1^{(1)}}-\overbrace{\xi_{11}}^{\gamma_1^{(1)}}x_{1;t-1}\right)^2}{2\underbrace{p_{11}^G}_{r_1^{(1)}}}\right\},$$

$$\exp\left\{-\frac{\left(\overbrace{x_{2;t}-z_2+g_{32}x_{3;t}-\xi_{22}x_{2;t-1}-\xi_{23}x_{3;t-1}}^{\beta_2^{(1)}}-\overbrace{\xi_{21}}^{\gamma_2^{(1)}}x_{1;t-1}\right)^2}{2\underbrace{p_{22}^G}_{r_2^{(1)}}}\right\},$$

$$\exp\left\{-\frac{\left(\overbrace{x_{3;t}-z_3-\xi_{32}x_{2;t-1}-\xi_{33}x_{3;t-1}}^{\beta_3^{(1)}}-\overbrace{\xi_{31}}^{\gamma_3^{(1)}}x_{1;t-1}\right)^2}{2\underbrace{p_{33}^G}_{r_3^{(1)}}}\right\},$$

$$\exp\left\{-\frac{\left(\overbrace{y_{1;t}-\rho_1+y_{2;t}l_{21}^{Rv}+y_{3;t}l_{31}^{Rv}-\alpha_{12}x_{2;t-1}-\alpha_{13}x_{3;t-1}}^{\beta_4^{(1)}}-\overbrace{\alpha_{11}}^{\gamma_4^{(1)}}x_{1;t-1}\right)^2}{2\underbrace{p_{11}^{Rv}}_{r_4^{(1)}}}\right\},$$

$$\exp\left\{-\frac{\left(\overbrace{y_{2;t}-\rho_2+y_{3;t}l_{32}^{Rv}-\alpha_{22}x_{2;t-1}-\alpha_{23}x_{3;t-1}}^{\beta_5^{(1)}}-\overbrace{\alpha_{21}}^{\gamma_5^{(1)}}x_{1;t-1}\right)^2}{2\underbrace{p_{22}^{Rv}}_{r_5}}\right\},$$

$$\exp\left\{-\frac{\left(\overbrace{y_{3;t}-\rho_3-\alpha_{32}x_{2;t-1}-\alpha_{33}x_{3;t-1}}^{\beta_6^{(1)}}-\overbrace{\alpha_{31}}^{\gamma_6^{(1)}}x_{1;t-1}\right)^2}{2\underbrace{p_{33}^{Rv}}_{r_6^{(1)}}}\right\},$$

$$\exp\left\{-\frac{\left(\overbrace{\hat{\mu}_{1;t-1}-l_{21}x_{2;t-1}-l_{31}x_{3;t-1}}^{\beta_7^{(1)}}-\overbrace{1}^{\gamma_7^{(1)}}\cdot x_{1;t-1}\right)^2}{2\underbrace{\hat{p}_{1;t-1}}_{r_7^{(1)}}}\right\}dx_{1;t-1}. \quad (2.60)$$

The upper right index (1) in (2.60) corresponds to the index of the factor $x_{1;t-1}$, being integrated out. Vector $[\beta_1^{(1)}, \ldots, \beta_{\mathring{\beta}=7}^{(1)}]'$ composes $\beta^{(1)}$, which is used in (2.29). Similarly, coefficients $\gamma_1^{(1)}, \ldots, \gamma_7^{(1)}$ determine the column vector $\gamma^{(1)}$, while diagonal matrix $\omega^{(1)} = \mathrm{diag}\left[{r_1^{(1)}}^{-1}, \ldots, {r_7^{(1)}}^{-1}\right]$. The capacity $\mathring{\beta} = 7$ is composed from $\mathring{x} + \mathring{y} + 1$, where the additional unit appears because of the Gaussian pdf of $x_{1;t-1}$.

After multiplication of vector $\beta^{(1)}$ and triangular matrix $U^{(1)}$, obtained according to (2.35), the new vector ${\beta^{(1)}}'{U^{(1)}}'$ (or transposed $U^{(1)}\beta^{(1)}$) takes the following form.

$$[\beta_1^{(1)}+\beta_2^{(1)}U_{21}^{(1)}+\ldots+\beta_7^{(1)}U_{71}^{(1)} \quad \beta_2^{(1)}+\beta_3^{(1)}U_{32}^{(1)}+\ldots+\beta_7^{(1)}U_{72}^{(1)} \quad \beta_3^{(1)}+\beta_4^{(1)}U_{43}^{(1)}+\ldots+\beta_7^{(1)}U_{73}^{(1)} \quad \ldots \quad \beta_7^{(1)}].$$

Multiplication $({\beta^{(1)}}'{U^{(1)}}')D^{(1)}(U^{(1)}\beta^{(1)})$ gives the result in the form, written in (2.30).

2.4.5 Integration over the second state factor

Integration of (2.45) over the factor $x_{2;t-1}$ is fulfilled similarly, according to (2.29) with index $m = 2$. The capacity $\mathring{\beta}^{(m)} = 8$ of vector $\beta^{(m)}$ is now composed from $\mathring{x} + \mathring{y} + 1 + 1$. Two additional units appear because of remained from the previous integration $\beta_7^{(1)}$, containing $x_{2;t-1}$, and Gaussian pdf of factor $x_{2;t-1}$, being integrated out. It is clear, that vector $\beta^{(2)}$ is determined as $[\beta_1^{(2)}, \ldots, \beta_8^{(2)}]'$, with $\beta_k^{(2)}$, $k = \{1, \ldots, 8\}$, which denotes the sum of all members but $x_{2;t-1}$ in the quadratic form of the k-th Gaussian pdf in (2.45). Coefficients $\gamma_k^{(2)}$, corresponding to $x_{2;t-1}$ in the k-th Gaussian pdf in (2.45), compose vector $\gamma^{(2)} \equiv [\gamma_1^{(2)}, \ldots, \gamma_8^{(2)}]'$ to be used in (2.30). Diagonal matrix $\omega^{(2)}$ is defined as $\mathrm{diag}\left[{r_1^{(2)}}^{-1}, \ldots, {r_8^{(2)}}^{-1}\right]$, where $r_k^{(2)} = \frac{1}{D_{kk}^{(1)}}, k = \{1, \ldots, 7\}$ from previous integration, while $r_8^{(2)} = \hat{p}_{2;t-1}$ from (2.48). According to recursions, defined for (2.36-2.43), the vectors $\beta^{(2)} \equiv [\beta_1^{(2)}, \ldots, \beta_8^{(2)}]'$, $\gamma^{(2)} \equiv [\gamma_1^{(2)}, \ldots, \gamma_8^{(2)}]'$ and diagonal matrix $\omega^{(2)}$ include now the following components.

$$\exp\left\{ -\frac{\left(\overbrace{x_{1;t} - \mu_{1;t}^{(1)} + g_{21}^{(1)} x_{2;t} + g_{31}^{(1)} x_{3;t} - \xi_{13}^{(1)} x_{3;t-1} + \eta_{11}^{(1)} y_{1;t} + \eta_{21}^{(1)} y_{2;t} + \eta_{31}^{(1)} y_{3;t}}^{\beta_1^{(2)}} - \overbrace{\xi_{12}^{(1)}}^{\gamma_1^{(2)}} x_{2;t-1}\right)^2}{2 \underbrace{\frac{1}{D_{11}^{(1)}}}_{r_1^{(2)}}} \right\}, \tag{2.61}$$

where

$$\begin{aligned}
\mu_{1;t}^{(1)} &= z_1 + z_2 U_{21}^{(1)} + z_3 U_{31}^{(1)} + \rho_1 U_{41}^{(1)} + \rho_2 U_{51}^{(1)} + \rho_3 U_{61}^{(1)} - \hat{\mu}_{1;t-1} U_{71}^{(1)}, \\
g_{21}^{(1)} &= g_{21} + U_{21}^{(1)}, \\
g_{31}^{(1)} &= g_{31} + g_{32} U_{21}^{(1)} + U_{31}^{(1)}, \\
\xi_{12}^{(1)} &= \xi_{12} + \xi_{22} U_{21}^{(1)} + \xi_{32} U_{31}^{(1)} + \alpha_{12} U_{41}^{(1)} + \alpha_{22} U_{51}^{(1)} + \alpha_{32} U_{61}^{(1)} + l_{21} U_{71}^{(1)}, \\
\xi_{13}^{(1)} &= \xi_{13} + \xi_{23} U_{21}^{(1)} + \xi_{33} U_{31}^{(1)} + \alpha_{13} U_{41}^{(1)} + \alpha_{23} U_{51}^{(1)} + \alpha_{33} U_{61}^{(1)} + l_{31} U_{71}^{(1)}, \\
\eta_{11}^{(1)} &= U_{41}^{(1)}, \\
\eta_{21}^{(1)} &= l_{21}^{Rv} U_{41}^{(1)} + U_{51}^{(1)}, \\
\eta_{31}^{(1)} &= l_{31}^{Rv} U_{41}^{(1)} + l_{32}^{Rv} U_{51}^{(1)} + U_{61}^{(1)},
\end{aligned}$$

$$\exp\left\{ -\frac{\left(\overbrace{x_{2;t} - \mu_{2;t}^{(1)} + g_{32}^{(1)} x_{3;t} - \xi_{23}^{(1)} x_{3;t-1} + \eta_{12}^{(1)} y_{1;t} + \eta_{22}^{(1)} y_{2;t} + \eta_{32}^{(1)} y_{3;t}}^{\beta_2^{(2)}} - \overbrace{\xi_{22}^{(1)}}^{\gamma_2^{(2)}} x_{2;t-1}\right)^2}{2 \underbrace{\frac{1}{D_{22}^{(1)}}}_{r_2^{(2)}}} \right\}, \tag{2.62}$$

where

$$
\begin{aligned}
\mu_{2;t}^{(1)} &= z_2 + z_3 U_{32}^{(1)} + \rho_1 U_{42}^{(1)} + \rho_2 U_{52}^{(1)} + \rho_3 U_{62}^{(1)} - \hat{\mu}_{1;t-1} U_{72}^{(1)}, \\
g_{32}^{(1)} &= g_{32} + U_{32}^{(1)}, \\
\xi_{22}^{(1)} &= \xi_{22} + \xi_{32} U_{32}^{(1)} + \alpha_{12} U_{42}^{(1)} + \alpha_{22} U_{52}^{(1)} + \alpha_{32} U_{62}^{(1)} + l_{21} U_{72}^{(1)}, \\
\xi_{23}^{(1)} &= \xi_{23} + \xi_{33} U_{32}^{(1)} + \alpha_{13} U_{42}^{(1)} + \alpha_{23} U_{52}^{(1)} + \alpha_{33} U_{62}^{(1)} + l_{31} U_{72}^{(1)}, \\
\eta_{12}^{(1)} &= U_{42}^{(1)}, \\
\eta_{22}^{(1)} &= l_{21}^{Rv} U_{42}^{(1)} + U_{52}^{(1)}, \\
\eta_{32}^{(1)} &= l_{31}^{Rv} U_{42}^{(1)} + l_{32}^{Rv} U_{52}^{(1)} + U_{62}^{(1)},
\end{aligned}
$$

$$
\exp\left\{ -\frac{\left(\overbrace{x_{3;t} - \mu_{3;t}^{(1)} - \xi_{33}^{(1)} x_{3;t-1} + \eta_{13}^{(1)} y_{1;t} + \eta_{23}^{(1)} y_{2;t} + \eta_{33}^{(1)} y_{3;t}}^{\beta_3^{(2)}} - \overbrace{\xi_{32}^{(1)}}^{\gamma_3^{(2)}} x_{2;t-1} \right)^2}{2 \underbrace{\frac{1}{D_{33}^{(1)}}}_{r_3^{(2)}}} \right\}, \tag{2.63}
$$

where

$$
\begin{aligned}
\mu_{3;t}^{(1)} &= z_3 + \rho_1 U_{43}^{(1)} + \rho_2 U_{53}^{(1)} + \rho_3 U_{63}^{(1)} - \hat{\mu}_{1;t-1} U_{73}^{(1)}, \\
\xi_{32}^{(1)} &= \xi_{32} + \alpha_{12} U_{43}^{(1)} + \alpha_{22} U_{53}^{(1)} + \alpha_{32} U_{63}^{(1)} + l_{21} U_{73}^{(1)}, \\
\xi_{33}^{(1)} &= \xi_{33} + \alpha_{13} U_{43}^{(1)} + \alpha_{23} U_{53}^{(1)} + \alpha_{33} U_{63}^{(1)} + l_{31} U_{73}^{(1)}, \\
\eta_{13}^{(1)} &= U_{43}^{(1)}, \\
\eta_{23}^{(1)} &= l_{21}^{Rv} U_{43}^{(1)} + U_{53}^{(1)}, \\
\eta_{33}^{(1)} &= l_{31}^{Rv} U_{43}^{(1)} + l_{32}^{Rv} U_{53}^{(1)} + U_{63}^{(1)},
\end{aligned}
$$

$$
\exp\left\{ -\frac{\left(\overbrace{y_{1;t} - \rho_1^{(1)} + l_{21}^{Rv\,(1)} y_{2;t} + l_{21}^{Rv\,(1)} y_{3;t} - \alpha_{13}^{(1)} x_{3;t-1}}^{\beta_4^{(2)}} - \overbrace{\alpha_{12}^{(1)}}^{\gamma_4^{(2)}} x_{2;t-1} \right)^2}{2 \underbrace{\frac{1}{D_{44}^{(1)}}}_{r_4^{(2)}}} \right\}, \tag{2.64}
$$

where

$$
\begin{aligned}
\rho_1^{(1)} &= \rho_1 + \rho_2 U_{54}^{(1)} + \rho_3 U_{64}^{(1)} - \hat{\mu}_{1;t-1} U_{74}^{(1)}, \\
l_{21}^{Rv\,(1)} &= l_{21}^{Rv} + U_{54}^{(1)}, \\
l_{31}^{Rv\,(1)} &= l_{31}^{Rv} + l_{32}^{Rv} U_{54}^{(1)} + U_{64}^{(1)}, \\
\alpha_{12}^{(1)} &= \alpha_{12} + \alpha_{22} U_{54}^{(1)} + \alpha_{32} U_{64}^{(1)} + l_{21} U_{74}^{(1)}, \\
\alpha_{13}^{(1)} &= \alpha_{13} + \alpha_{23} U_{54}^{(1)} + \alpha_{33} U_{64}^{(1)} + l_{31} U_{74}^{(1)},
\end{aligned}
$$

$$\exp\left\{-\frac{\left(\overbrace{y_{2;t}-\rho_2^{(1)}+l_{32}^{Rv\,(1)}y_{3;t}-\alpha_{23}^{(1)}x_{3;t-1}}^{\beta_5^{(2)}}-\overbrace{\alpha_{22}^{(1)}}^{\gamma_5^{(2)}}x_{2;t-1}\right)^2}{2\underbrace{\frac{1}{D_{55}^{(1)}}}_{r_5^{(2)}}}\right\}, \tag{2.65}$$

where

$$\begin{aligned}
\rho_2^{(1)} &= \rho_2+\rho_3U_{65}^{(1)}-\hat{\mu}_{1;t-1}U_{75}^{(1)},\\
l_{32}^{Rv\,(1)} &= l_{32}^{Rv}+U_{65}^{(1)},\\
\alpha_{22}^{(1)} &= \alpha_{22}+\alpha_{32}U_{65}^{(1)}+l_{21}U_{75}^{(1)},\\
\alpha_{23}^{(1)} &= \alpha_{23}+\alpha_{33}U_{65}^{(1)}+l_{31}U_{75}^{(1)},
\end{aligned}$$

$$\exp\left\{-\frac{\left(\overbrace{y_{3;t}-\rho_3^{(1)}-\alpha_{33}^{(1)}x_{3;t-1}}^{\beta_6^{(2)}}-\overbrace{\alpha_{32}^{(1)}}^{\gamma_6^{(2)}}x_{2;t-1}\right)^2}{2\underbrace{\frac{1}{D_{66}^{(1)}}}_{r_6^{(2)}}}\right\}, \tag{2.66}$$

where

$$\begin{aligned}
\rho_3^{(1)} &= \rho_3-\hat{\mu}_{1;t-1}U_{76}^{(1)},\\
\alpha_{32}^{(1)} &= \alpha_{32}+l_{21}U_{76}^{(1)},\\
\alpha_{33}^{(1)} &= \alpha_{33}+l_{31}U_{76}^{(1)},
\end{aligned}$$

$$\exp\left\{-\frac{\left(\overbrace{\hat{\mu}_{1;t-1}-l_{31}x_{3;t-1}}^{\beta_7^{(2)}}-\overbrace{l_{21}}^{\gamma_7^{(2)}}x_{2;t-1}\right)^2}{2\underbrace{\frac{1}{D_{77}^{(1)}}}_{r_7^{(2)}}}\right\}, \tag{2.67}$$

and, finally,

$$\exp\left\{-\frac{\left(\overbrace{\hat{\mu}_{2;t-1}-l_{32}x_{3;t-1}}^{\beta_8^{(2)}}-\overbrace{1}^{\gamma_8^{(2)}}\cdot x_{2;t-1}\right)^2}{2\underbrace{\hat{p}_{2;t-1}}_{r_8^{(2)}}}\right\}. \tag{2.68}$$

2.4.6 Integration over the third state factor

According to (2.29) with index $m=3$, integration of (2.45) over the factor $x_{3;t-1}$ is fulfilled in the similar way. The capacity $\mathring{\beta}^{(m)}=9$ of vector $\beta^{(m)}$ is now composed from $\mathring{x}+\mathring{y}+1+1+1$, with all remainders, containing factor

$x_{3;t-1}$ and its pdf $f\left(x_{3;t-1}\middle| d^{1:t-1}\right)$. Vector $\beta^{(3)}$ is defined as $[\beta_1^{(3)},\ldots,\beta_9^{(3)}]'$, with $\beta_k^{(3)}$, $k=\{1,\ldots,9\}$, which denotes the sum of all members but $x_{3;t-1}$ in the quadratic form of the k-th Gaussian pdf in (2.45). Coefficients $\gamma_k^{(3)}$, corresponding to $x_{3;t-1}$ in the k-th Gaussian pdf in (2.45), compose vector $\gamma^{(3)}\equiv[\gamma_1^{(3)},\ldots,\gamma_9^{(3)}]'$ to be used in (2.30). Diagonal matrix $\omega^{(3)}$ is defined as diag $\left[{r_1^{(3)}}^{-1},\ldots,{r_9^{(3)}}^{-1}\right]$, where $r_k^{(3)}=\frac{1}{D_{kk}^{(2)}}, k=\{1,\ldots,8\}$ from previous integration, while $r_9^{(3)}=\hat{p}_{3;t-1}$ from (2.49). According to recursions (2.36-2.43), the product of the following pdfs should be integrated over $x_{3;t-1}$ now.

$$\exp\left\{-\frac{\left(\overbrace{x_{1;t}-\mu_{1;t}^{(2)}+g_{21}^{(2)}x_{2;t}+g_{31}^{(2)}x_{3;t}+\eta_{11}^{(2)}y_{1;t}+\eta_{21}^{(2)}y_{2;t}+\eta_{31}^{(2)}y_{3;t}}^{\beta_1^{(3)}}-\overbrace{\xi_{13}^{(2)}}^{\gamma_1^{(3)}}x_{3;t-1}\right)^2}{2\underbrace{\frac{1}{D_{11}^{(2)}}}_{r_1^{(3)}}}\right\}, \tag{2.69}$$

where

$$\begin{aligned}
\mu_{1;t}^{(2)} &= \mu_{1;t}^{(1)}+\mu_{2;t}^{(1)}U_{21}^{(2)}+\mu_{3;t}^{(1)}U_{31}^{(2)}+\rho_1^{(1)}U_{41}^{(2)}+\rho_2^{(1)}U_{51}^{(2)}+\rho_3^{(1)}U_{61}^{(2)}-\hat{\mu}_{1;t-1}U_{71}^{(2)}-\hat{\mu}_{2;t-1}U_{81}^{(2)},\\
g_{21}^{(2)} &= g_{21}^{(1)}+U_{21}^{(2)},\\
g_{31}^{(2)} &= g_{31}^{(1)}+g_{32}^{(1)}U_{21}^{(2)}+U_{31}^{(2)},\\
\eta_{11}^{(2)} &= \eta_{11}^{(1)}+\eta_{12}^{(1)}U_{21}^{(2)}+\eta_{13}^{(1)}U_{31}^{(2)}+U_{41}^{(2)},\\
\eta_{21}^{(2)} &= \eta_{21}^{(1)}+\eta_{22}^{(1)}U_{21}^{(2)}+\eta_{23}^{(1)}U_{31}^{(2)}+{l_{21}^{Rv}}^{(1)}U_{41}^{(2)}+U_{51}^{(2)},\\
\eta_{31}^{(2)} &= \eta_{31}^{(1)}+\eta_{32}^{(1)}U_{21}^{(2)}+\eta_{33}^{(1)}U_{31}^{(2)}+{l_{31}^{Rv}}^{(1)}U_{41}^{(2)}+{l_{32}^{Rv}}^{(1)}U_{51}^{(2)}+U_{61}^{(2)},\\
\xi_{13}^{(2)} &= \xi_{13}^{(1)}+\xi_{23}^{(1)}U_{21}^{(2)}+\xi_{33}^{(1)}U_{31}^{(2)}+\alpha_{13}^{(1)}U_{41}^{(2)}+\alpha_{23}^{(1)}U_{51}^{(2)}+\alpha_{33}^{(1)}U_{61}^{(2)}+l_{31}U_{71}^{(2)}+l_{32}U_{81}^{(2)},
\end{aligned}$$

$$\exp\left\{-\frac{\left(\overbrace{x_{2;t}-\mu_{2;t}^{(2)}+g_{32}^{(2)}x_{3;t}+\eta_{12}^{(2)}y_{1;t}+\eta_{22}^{(2)}y_{2;t}+\eta_{32}^{(2)}y_{3;t}}^{\beta_2^{(3)}}-\overbrace{\xi_{23}^{(2)}}^{\gamma_2^{(3)}}x_{3;t-1}\right)^2}{2\underbrace{\frac{1}{D_{22}^{(2)}}}_{r_2^{(3)}}}\right\}, \tag{2.70}$$

where

$$\begin{aligned}
\mu_{2;t}^{(2)} &= \mu_{2;t}^{(1)}+\mu_{3;t}^{(1)}U_{32}^{(2)}+\rho_1^{(1)}U_{42}^{(2)}+\rho_2^{(1)}U_{52}^{(2)}+\rho_3^{(1)}U_{62}^{(2)}-\hat{\mu}_{1;t-1}U_{72}^{(2)}-\hat{\mu}_{2;t-1}U_{82}^{(2)},\\
g_{32}^{(2)} &= g_{32}^{(1)}+U_{32}^{(2)},\\
\eta_{12}^{(2)} &= \eta_{12}^{(1)}+\eta_{13}^{(1)}U_{32}^{(2)}+U_{42}^{(2)},\\
\eta_{22}^{(2)} &= \eta_{22}^{(1)}+\eta_{23}^{(1)}U_{32}^{(2)}+{l_{21}^{Rv}}^{(1)}U_{42}^{(2)}+U_{52}^{(2)},\\
\eta_{32}^{(2)} &= \eta_{32}^{(1)}+\eta_{33}^{(1)}U_{32}^{(2)}+{l_{31}^{Rv}}^{(1)}U_{42}^{(2)}+{l_{32}^{Rv}}^{(1)}U_{52}^{(2)}+U_{62}^{(2)},\\
\xi_{23}^{(2)} &= \xi_{23}^{(1)}+\xi_{33}^{(1)}U_{32}^{(2)}+\alpha_{13}^{(1)}U_{42}^{(2)}+\alpha_{23}^{(1)}U_{52}^{(2)}+\alpha_{33}^{(1)}U_{62}^{(2)}+l_{31}U_{72}^{(2)}+l_{32}U_{82}^{(2)},
\end{aligned}$$

$$\exp\left\{-\frac{\left(\overbrace{x_{3;t}-\mu_{3;t}^{(2)}+\eta_{13}^{(2)}y_{1;t}+\eta_{23}^{(2)}y_{2;t}+\eta_{33}^{(2)}y_{3;t}}^{\beta_3^{(3)}}-\overbrace{\xi_{33}^{(2)}}^{\gamma_3^{(3)}}x_{3;t-1}\right)^2}{2\underbrace{\frac{1}{D_{33}^{(2)}}}_{r_3^{(3)}}}\right\}, \tag{2.71}$$

where

$$\begin{aligned}
\mu_{3;t}^{(2)} &= \mu_{3;t}^{(1)} + \rho_1^{(1)} U_{43}^{(2)} + \rho_2^{(1)} U_{53}^{(2)} + \rho_3^{(1)} U_{63}^{(2)} - \hat{\mu}_{1;t-1} U_{73}^{(2)} - \hat{\mu}_{2;t-1} U_{83}^{(2)}, \\
\eta_{13}^{(2)} &= \eta_{13}^{(1)} + U_{43}^{(2)}, \\
\eta_{23}^{(2)} &= \eta_{23}^{(1)} + l_{21}^{Rv\,(1)} U_{43}^{(2)} + U_{53}^{(2)}, \\
\eta_{33}^{(2)} &= \eta_{33}^{(1)} + l_{31}^{Rv\,(1)} U_{43}^{(2)} + l_{32}^{Rv\,(1)} U_{53}^{(2)} + U_{63}^{(2)}, \\
\xi_{33}^{(2)} &= \xi_{33}^{(1)} + \alpha_{13}^{(1)} U_{43}^{(2)} + \alpha_{23}^{(1)} U_{53}^{(2)} + \alpha_{33}^{(1)} U_{63}^{(2)} + l_{31} U_{73}^{(2)} + l_{32} U_{83}^{(2)},
\end{aligned}$$

$$\exp\left\{ -\frac{\left(\overbrace{y_{1;t} - \rho_1^{(2)} + l_{21}^{Rv\,(2)} y_{2;t} + l_{31}^{Rv\,(2)} y_{3;t}}^{\beta_4^{(3)}} - \overbrace{\alpha_{13}^{(2)}}^{\gamma_4^{(3)}} x_{3;t-1} \right)^2}{2 \underbrace{\frac{1}{D_{44}^{(2)}}}_{r_4^{(3)}}} \right\}, \tag{2.72}$$

where

$$\begin{aligned}
\rho_1^{(2)} &= \rho_1^{(1)} + \rho_2^{(1)} U_{54}^{(2)} + \rho_3^{(1)} U_{64}^{(2)} - \hat{\mu}_{1;t-1} U_{74}^{(2)} - \hat{\mu}_{2;t-1} U_{84}^{(2)}, \\
l_{21}^{Rv\,(2)} &= l_{21}^{Rv\,(1)} + U_{54}^{(2)}, \\
l_{31}^{Rv\,(2)} &= l_{31}^{Rv\,(1)} + l_{32}^{Rv\,(1)} U_{54}^{(2)} + U_{64}^{(2)}, \\
\alpha_{13}^{(2)} &= \alpha_{13}^{(1)} + \alpha_{23}^{(1)} U_{54}^{(2)} + \alpha_{33}^{(1)} U_{64}^{(2)} + l_{31} U_{74}^{(2)} + l_{32} U_{84}^{(2)},
\end{aligned}$$

$$\exp\left\{ -\frac{\left(\overbrace{y_{2;t} - \rho_2^{(2)} + l_{32}^{Rv\,(2)} y_{3;t}}^{\beta_5^{(3)}} - \overbrace{\alpha_{23}^{(2)}}^{\gamma_5^{(3)}} x_{3;t-1} \right)^2}{2 \underbrace{\frac{1}{D_{55}^{(2)}}}_{r_5^{(3)}}} \right\}, \tag{2.73}$$

where

$$\begin{aligned}
\rho_2^{(2)} &= \rho_2^{(1)} + \rho_3^{(1)} U_{65}^{(2)} - \hat{\mu}_{1;t-1} U_{75}^{(2)} - \hat{\mu}_{2;t-1} U_{85}^{(2)}, \\
l_{32}^{Rv\,(2)} &= l_{32}^{Rv\,(1)} + U_{65}^{(2)}, \\
\alpha_{23}^{(2)} &= \alpha_{23}^{(1)} + \alpha_{33}^{(1)} U_{65}^{(2)} + l_{31} U_{75}^{(2)} + l_{32} U_{85}^{(2)},
\end{aligned}$$

$$\exp\left\{ -\frac{\left(\overbrace{y_{3;t} - \rho_3^{(2)}}^{\beta_6^{(3)}} - \overbrace{\alpha_{33}^{(2)}}^{\gamma_6^{(3)}} x_{3;t-1} \right)^2}{2 \underbrace{\frac{1}{D_{66}^{(2)}}}_{r_6^{(3)}}} \right\}, \tag{2.74}$$

where

$$\begin{aligned}
\rho_3^{(2)} &= \rho_3^{(1)} - \hat{\mu}_{1;t-1} U_{76}^{(2)} - \hat{\mu}_{2;t-1} U_{86}^{(2)}, \\
\alpha_{33}^{(2)} &= \alpha_{33}^{(1)} + l_{31} U_{76}^{(2)} + l_{32} U_{86}^{(2)},
\end{aligned}$$

and finally, three last ones,

$$\exp\left\{-\frac{\left(\overbrace{\hat{\mu}_{1;t-1}}^{\beta_7^{(3)}}-\overbrace{l_{31}}^{\gamma_7^{(3)}}x_{3;t-1}\right)^2}{2\underbrace{\frac{1}{D_{77}^{(2)}}}_{r_7^{(3)}}}\right\}, \tag{2.75}$$

$$\exp\left\{-\frac{\left(\overbrace{\hat{\mu}_{2;t-1}}^{\beta_8^{(3)}}-\overbrace{l_{32}}^{\gamma_8^{(3)}}x_{3;t-1}\right)^2}{2\underbrace{\frac{1}{D_{88}^{(2)}}}_{r_8^{(3)}}}\right\}, \tag{2.76}$$

$$\exp\left\{-\frac{\left(\overbrace{\hat{\mu}_{3;t-1}}^{\beta_9^{(3)}}-\overbrace{1}^{\gamma_9^{(3)}}\cdot x_{3;t-1}\right)^2}{2\underbrace{\hat{p}_{3;t-1}}_{r_9^{(3)}}}\right\}. \tag{2.77}$$

2.4.7 Final results of simultaneous data & time updating

After (final) integration over $x_{3;t-1}$ the distributions of Gaussian state and output factors have the following form according to 2.36-2.43).

$$\exp\left\{-\frac{\left(x_{1;t}-\mu_{1;t}^{(3)}+g_{21}^{(3)}x_{2;t}+g_{31}^{(3)}x_{3;t}+\eta_{11}^{(3)}y_{1;t}+\eta_{21}^{(3)}y_{2;t}+\eta_{31}^{(3)}y_{3;t}\right)^2}{2\frac{1}{D_{11}^{(3)}}}\right\}, \tag{2.78}$$

where

$$\begin{aligned}
\mu_{1;t}^{(3)} &= \mu_{1;t}^{(2)}+\mu_{2;t}^{(2)}U_{21}^{(3)}+\mu_{3;t}^{(2)}U_{31}^{(3)}+\rho_1^{(2)}U_{41}^{(3)}+\rho_2^{(2)}U_{51}^{(3)}+\rho_3^{(2)}U_{61}^{(3)}-\hat{\mu}_{1;t-1}U_{71}^{(3)}-\hat{\mu}_{2;t-1}U_{81}^{(3)}-\hat{\mu}_{3;t-1}U_{91}^{(3)},\\
g_{21}^{(3)} &= g_{21}^{(2)}+U_{21}^{(3)},\\
g_{31}^{(3)} &= g_{31}^{(2)}+g_{32}^{(2)}U_{21}^{(3)}+U_{31}^{(3)},\\
\eta_{11}^{(3)} &= \eta_{11}^{(2)}+\eta_{12}^{(2)}U_{21}^{(3)}+\eta_{13}^{(2)}U_{31}^{(3)}+U_{41}^{(3)},\\
\eta_{21}^{(3)} &= \eta_{21}^{(2)}+\eta_{22}^{(2)}U_{21}^{(3)}+\eta_{23}^{(2)}U_{31}^{(3)}+l_{21}^{Rv\,(2)}U_{41}^{(3)}+U_{51}^{(3)},\\
\eta_{31}^{(3)} &= \eta_{31}^{(2)}+\eta_{32}^{(2)}U_{21}^{(3)}+\eta_{33}^{(2)}U_{31}^{(3)}+l_{31}^{Rv\,(2)}U_{41}^{(3)}+l_{32}^{Rv\,(2)}U_{51}^{(3)}+U_{61}^{(3)},
\end{aligned}$$

$$\exp\left\{-\frac{\left(x_{2;t}-\mu_{2;t}^{(3)}+g_{32}^{(3)}x_{3;t}+\eta_{12}^{(3)}y_{1;t}+\eta_{22}^{(3)}y_{2;t}+\eta_{32}^{(3)}y_{3;t}\right)^2}{2\frac{1}{D_{22}^{(3)}}}\right\}, \tag{2.79}$$

where

$$\begin{aligned}
\mu_{2;t}^{(3)} &= \mu_{2;t}^{(2)}+\mu_{3;t}^{(2)}U_{32}^{(3)}+\rho_1^{(2)}U_{42}^{(3)}+\rho_2^{(2)}U_{52}^{(3)}+\rho_3^{(2)}U_{62}^{(3)}-\hat{\mu}_{1;t-1}U_{72}^{(3)}-\hat{\mu}_{2;t-1}U_{82}^{(3)}-\hat{\mu}_{3;t-1}U_{92}^{(3)},\\
g_{32}^{(3)} &= g_{32}^{(2)}+U_{32}^{(3)},\\
\eta_{12}^{(3)} &= \eta_{12}^{(2)}+\eta_{13}^{(2)}U_{32}^{(3)}+U_{42}^{(3)},\\
\eta_{22}^{(3)} &= \eta_{22}^{(2)}+\eta_{23}^{(2)}U_{32}^{(3)}+l_{21}^{Rv\,(2)}U_{42}^{(3)}+U_{52}^{(3)},\\
\eta_{32}^{(3)} &= \eta_{32}^{(2)}+\eta_{33}^{(2)}U_{32}^{(3)}+l_{31}^{Rv\,(2)}U_{42}^{(3)}+l_{32}^{Rv\,(2)}U_{52}^{(3)}+U_{62}^{(3)},
\end{aligned}$$

$$\exp\left\{-\frac{\left(x_{3;t}-\mu_{3;t}^{(3)}+\eta_{13}^{(3)}y_{1;t}+\eta_{23}^{(3)}y_{2;t}+\eta_{33}^{(3)}y_{3;t}\right)^2}{2\frac{1}{D_{33}^{(3)}}}\right\}, \tag{2.80}$$

where

$$\begin{aligned}
\mu_{3;t}^{(3)} &= \mu_{3;t}^{(2)}+\rho_1^{(2)}U_{43}^{(3)}+\rho_2^{(2)}U_{53}^{(3)}+\rho_3^{(2)}U_{63}^{(3)}-\hat{\mu}_{1;t-1}U_{73}^{(3)}-\hat{\mu}_{2;t-1}U_{83}^{(3)}-\hat{\mu}_{3;t-1}U_{93}^{(3)},\\
\eta_{13}^{(3)} &= \eta_{13}^{(2)}+U_{43}^{(3)},\\
\eta_{23}^{(3)} &= \eta_{23}^{(2)}+l_{21}^{Rv\,(2)}U_{43}^{(3)}+U_{53}^{(3)},\\
\eta_{33}^{(3)} &= \eta_{33}^{(2)}+l_{31}^{Rv\,(2)}U_{43}^{(3)}+l_{32}^{Rv\,(2)}U_{53}^{(3)}+U_{63}^{(3)},
\end{aligned}$$

$$\exp\left\{-\frac{\left(y_{1;t}-\rho_1^{(3)}+l_{21}^{Rv\,(3)}y_{2;t}+l_{31}^{Rv\,(3)}y_{3;t}\right)^2}{2\frac{1}{D_{44}^{(3)}}}\right\}, \tag{2.81}$$

where

$$\begin{aligned}
\rho_1^{(3)} &= \rho_1^{(2)}+\rho_2^{(2)}U_{54}^{(3)}+\rho_3^{(2)}U_{64}^{(3)}-\hat{\mu}_{1;t-1}U_{74}^{(3)}-\hat{\mu}_{2;t-1}U_{84}^{(3)}-\hat{\mu}_{3;t-1}U_{94}^{(3)},\\
l_{21}^{Rv\,(3)} &= l_{21}^{Rv\,(2)}+U_{54}^{(3)},\\
l_{31}^{Rv\,(3)} &= l_{31}^{Rv\,(2)}+l_{32}^{Rv\,(2)}U_{54}^{(3)}+U_{64}^{(3)},
\end{aligned}$$

$$\exp\left\{-\frac{\left(y_{2;t}-\rho_2^{(3)}+l_{32}^{Rv\,(3)}y_{3;t}\right)^2}{2\frac{1}{D_{55}^{(3)}}}\right\}, \tag{2.82}$$

where

$$\begin{aligned}
\rho_2^{(3)} &= \rho_2^{(2)}+\rho_3^{(2)}U_{65}^{(3)}-\hat{\mu}_{1;t-1}U_{75}^{(3)}-\hat{\mu}_{2;t-1}U_{85}^{(3)}-\hat{\mu}_{3;t-1}U_{95}^{(3)},\\
l_{32}^{Rv\,(3)} &= l_{32}^{Rv\,(2)}+U_{65}^{(3)},
\end{aligned}$$

and

$$\exp\left\{-\frac{\left(y_{3;t}-\rho_3^{(3)}\right)^2}{2\frac{1}{D_{66}^{(3)}}}\right\},\ \text{with}\ \ \rho_3^{(3)}=\rho_3^{(2)}-\hat{\mu}_{1;t-1}U_{76}^{(3)}-\hat{\mu}_{2;t-1}U_{86}^{(3)}-\hat{\mu}_{3;t-1}U_{96}^{(3)}. \tag{2.83}$$

The remained constants are out of interest and are considered only as proportional values.

$$\exp\left\{-\frac{\left(\overbrace{\hat{\mu}_{1;t-1}}^{\beta_7^{(3)}}+\hat{\mu}_{2;t-1}U_{87}^{(3)}+\hat{\mu}_{3;t-1}U_{97}^{(3)}\right)^2}{2\frac{1}{D_{77}^{(3)}}}\right\}, \tag{2.84}$$

$$\exp\left\{-\frac{\left(\overbrace{\hat{\mu}_{2;t-1}}^{\beta_8^{(3)}}+\hat{\mu}_{3;t-1}U_{98}^{(3)}\right)^2}{2\frac{1}{D_{88}^{(3)}}}\right\}, \tag{2.85}$$

$$\exp\left\{-\frac{\left(\overbrace{\hat{\mu}_{3;t-1}}^{\beta_9^{(3)}}\right)^2}{2\frac{1}{D_{99}^{(3)}}}\right\}. \tag{2.86}$$

2.5 Conclusion

The paper presents the modified algorithm of factorized Kalman filtering. A modification of the algorithm consists in simultaneous fulfillment of the data and time updating of the posterior factorized state estimate for system, whose state and output are described by the joint pdf. Such the modification enables to use multi-output normal observation model. The future work in this field includes the experiments with simulated traffic-control data.

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