Advances in Bayesian software reliability

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SOFTWARE RELIABILITY

Software reliability can be defined as the probability of failure-free operation of a computer code for a specified mission time in a specified input environment.
SOFTWARE RELIABILITY: MODELS

Seminal paper by Jelinski and Miranda (1972)

More than 100 models after it (Philip Boland, MMR2002)

Many models clustered in few classes

Search for unifying models (e.g. Self-exciting process, Chen and Singpurwalla, 1997)
SOFTWARE RELIABILITY: MODELS

Most software reliability models fall into one of two categories (Singpurwalla and Wilson, 1994)

- [Type I]: models on times between successive failures based on
  - [Type I-1] failure rates (e.g. Jelinski-Moranda)
  - [Type I-2] inter-failure times as function of previous inter-failure times (e.g. random coefficient autoregressive model, Singpurwalla and Soyer, 1985)

- [Type II] models (counting processes) on observed number of failures up to time \( t \) (e.g. NHPP)

Boland, MMR2002
SOFTWARE RELIABILITY: MODELS

Failures at $T_1, T_2, \ldots, T_n$

Inter-failure times $T_i - T_{i-1} \sim \mathcal{E}(\lambda_i)$, independent, $i = 1, \ldots, n$

- $\lambda_i = \phi(N - i + 1)$, $\phi \in \mathbb{R}^+$, $N \in \mathbb{N}$, (Jelinski-Moranda, 1972)
  - Program contains an initial number of bugs $N$
  - Each bug contributes the same amount to the failure rate
  - After each observed failure, a bug is detected and corrected

Straightforward Bayesian inference with priors $N \sim \mathcal{P}(\nu)$ and $\phi \sim \mathcal{G}(\alpha, \beta)$
SOFTWARE RELIABILITY: MODELS

• $\lambda_i = \phi(N - p(i - 1))$, $\phi \in \mathbb{R}^+$, $N \in \mathbb{N}$, $p \in [0, 1]$,
  (Goel-Okumoto, 1978)
  
  – Imperfect debugging

• $\lambda_i = \phi \delta^i$, $\phi \in \mathbb{R}^+$, $\delta \in (0, 1)$, (Moranda, 1975)
  
  – Failure rate (geometrically) decreasing

Failure rate constant between failures; different from next model
SOFTWARE RELIABILITY: MODELS

- \( N_t, t \geq 0 \) # events by time \( t \)
- \( N(y, s) \) # events in \((y, s]\)
- \( M(t) = \mathcal{E}N_t \) mean value function
- \( M(y, s) = M(s) - M(y) \) expected # events in \((y, s]\)

\( N_t, t \geq 0 \), NHPP with intensity function \( \lambda(t) \) iff

1. \( N_0 = 0 \)
2. independent increments
3. \( \mathcal{P}\{\# \text{ events in } (t, t + h) \geq 2\} = o(h) \)
4. \( \mathcal{P}\{\# \text{ events in } (t, t + h) = 1\} = \lambda(t)h + o(h) \)

\( \Rightarrow \mathcal{P}\{N(y, s) = k\} = \frac{M(y, s)^k}{k!}e^{-M(y, s)}, \forall k \in \mathcal{N} \)
SOFTWARE RELIABILITY: MODELS

\( \lambda(t) \equiv \lambda \ \forall t \Rightarrow \text{HPP} \)

- \( \lambda(t) \): intensity function of \( N_t \)

\( \lambda(t) := \lim_{\Delta \to 0} \frac{P\{N(t, t + \Delta] \geq 1\}}{\Delta}, \ \forall t \geq 0 \)

- \( \mu(t) := \frac{dM(t)}{dt} \): Rocof (rate of occurrence of failures)

Property 3. \( \Rightarrow \mu(t) = \lambda(t) \) a.e. \( \Rightarrow M(y, s) = \int_y^s \lambda(t)dt \)
SOFTWARE RELIABILITY: ENVIRONMENTS

Techniques to achieve reliable software systems, aimed at

- Fault prevention
- Fault removal
- Fault tolerance (i.e. providing service despite of faults)
- Fault forecasting (*room for statistics ...*)

Stages, including

- Testing (*e.g. decreasing $\lambda(t)$ in NHPP*)
- Operation (*e.g. constant $\lambda(t)$*)
- Debugging (*e.g. change in $\lambda(t)$*)

Michael R. Lyu
SOFTWARE RELIABILITY: ENVIRONMENTS

Software

- Desktop computing
- Client/server computing
- Web-deployed applications
- .net enterprise (e.g. banking on line)

Intervention

- Perfect repair
- Imperfect repair
- **Bugs introduction**

Different environments, different models
BAYESIAN SOFTWARE RELIABILITY

Kuo, 2005

Review paper

• Models

• Bayesian inference

• Model selection

• Optimal release policy
OUTLINE

• Statement of the problem

• Hidden Markov model

• Self-exciting process with latent variables

• Change points models

• Future research
STATEMENT OF THE PROBLEM

Bugs in software induce failures

Fixing current bugs sometimes implies introduction of new bugs

Lack of knowledge about effects of bugs fixing

⇒ need for models allowing for possible, unobserved introduction of new bugs in a context aimed to reduce bugs
**BUGS INTRODUCTION: MODELS**

Failures at $T_1, T_2, \ldots, T_n$

Inter-failure times $T_i - T_{i-1} \sim \mathcal{E}(\lambda_i)$, independent, $i = 1, \ldots, n$

- $\lambda_{i+1} = \lambda_i e^{-\theta_i}$, $\lambda_i, \theta_i \in \mathbb{R}^+$, independent
  
  *(Gaudoin, Lavergne and Soler, 1994)*

- $\theta_i = 0 \Rightarrow$ no debugging effect
- $\theta_i > 0 \Rightarrow$ good quality debugging
- $\theta_i < 0 \Rightarrow$ **bad quality debugging**
BUGS INTRODUCTION: MODELS

• \( \lambda_{i+1} = (1 - \alpha_i - \beta_i) \lambda_i + \mu \beta_i, \quad \lambda_i, \mu \in \mathbb{R}^+, \ (Gaudoin, 1999) \)
  
  – Imperfect debugging
  
  – \( \alpha_i \) good debugging rate
  
  – \( \beta_i \) bad debugging rate
Birth-death process (*Kremer, 1983*)

- $p_n(t) = \Pr\{X(t) = n\}$
- $\nu(t)$ birth rate
- $\mu(t)$ death rate
- $a$ initial population

\[ p'_n(t) = (n-1)\nu(t)p_{n-1}(t) - n[\nu(t) + \mu(t)]p_n(t) + (n+1)\mu(t)p_{n+1}(t), n \geq 0 \]

with $p_{-1} \equiv 0$ and $p_n(0) = 1_{\{n=a\}}$
HIDDEN MARKOV MODEL

Failure times $t_1 < t_2 < \ldots < t_n$ in $(0, y]$

$Y_t$ latent process describing reliability status of software at time $t$ (e.g. growing, decreasing and constant)

$Y_t$ may change only after a failure
$\Rightarrow Y_t = Y_m$ for $t \in (t_{m-1}, t_m]$, $m = 1, \ldots, n + 1$
with $t_0 = 0$, $t_{n+1} = y$ and $Y_{t_0} = Y_0$ given for now
$\Rightarrow$ consider $\{Y_n\}_{n \in \mathbb{N}}$ Markov chain with discrete state space $E$

$X_m$ interarrival time of $m$-th failure, $m = 1, \ldots, n$
HIDDEN MARKOV MODEL

Markov chain $Y = \{Y_n\}_{n \in \mathbb{N}}$

- $E$ discrete state space ($\text{card}(E) = k < \infty$)

- $\mathbb{P}$ transition matrix with rows $\mathbb{P}_i = (P_{i1}, \ldots, P_{ik})$, $i = 1, \ldots, k$

Interarrival times $X_m|Y_m = i \sim \mathcal{E}(\lambda(i))$, $i = 1, \ldots, k$, $m = 1, \ldots, n$

$\mathbb{P}$ and $\lambda(i)$ unknown
HIDDEN MARKOV MODEL

(Durand and Gaudoin, 2005)

Parameter estimation

- Data partially observed (only $X_m$ but not $Y_m$)  
  $\Rightarrow$ difficult parameter estimation by maximum likelihood

- $\Rightarrow$ EM algorithm for likelihood maximisation in the context of missing values (McLachlan and Krishnan, 1997)

- $\Rightarrow$ sequence of values converging to the consistent solution of the likelihood equation, **provided** the starting point is close to the optimal point

- $\Rightarrow$ start from many initial values
HIDDEN MARKOV MODEL

(Durand and Gaudoin, 2005)

Hidden states number estimation

• Take any $k \in [K_{\text{min}}, K_{\text{max}}]$

• For each $k$ compute MLE via EM algorithm

• Choose $k$ with lowest BIC
HIDDEN MARKOV MODEL

(Durand and Gaudoin, 2005)

Selection of transition matrix via BIC

E.g. ordered $\lambda(1) > \lambda(2) > \ldots > \lambda(k)$

- Upper triangular matrix $\Rightarrow$ failure rates can only decrease
- Tridiagonal matrix $\Rightarrow$ only small increase and decrease in failure rate
HIDDEN MARKOV MODEL

$X_m$'s independent given $Y \Rightarrow f(X_1, \ldots, X_n|Y) = \prod_{m=1}^{n} f(X_m|Y)$

$\mathbb{P}_i \sim \text{Dir} (\alpha_{i1}, \ldots, \alpha_{ik}), \forall i \in E$, i.e. $\pi(\mathbb{P}_i) \propto \prod_{j=1}^{k} P_{ij}^{\alpha_{ij}-1}$

Independent $\lambda(i) \sim \text{G}(a(i), b(i)), \forall i \in E$

Interest in posterior distribution of $\Theta = (\lambda^{(k)}, \mathbb{P}, Y^{(n)})$

- $\lambda^{(k)} = (\lambda(1), \ldots, \lambda(k))$

- $Y^{(n)} = (Y_1, \ldots, Y_n)$
LIKELIHOOD

For observed $Y$, likelihood given by

$$f(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = f(X_1, \ldots, X_n|Y_1, \ldots, Y_n)f(Y_1, \ldots, Y_n)$$

$$= \prod_{m=1}^{n} P_{Y_{m-1}Y_m}\lambda(Y_m)e^{-\lambda(Y_m)X_m}$$

Here unobserved $Y$ treated as parameter

$$\Rightarrow L(\Theta) = \prod_{m=1}^{n} P_{Y_{m-1}Y_m}\lambda(Y_m)e^{-\lambda(Y_m)X_m}$$

Posterior distribution $\pi(\Theta|X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ proportional to

$$\prod_{m=1}^{n} \left[ P_{Y_{m-1}Y_m}\lambda(Y_m)e^{-\lambda(Y_m)X_m} \right] \prod_{i=1}^{k} \left[ [\lambda(i)]^{a(i)-1}e^{-b(i)\lambda(i)} \prod_{j=1}^{k} P_{ij}^{\alpha_{ij}-1} \right]$$
FULL CONDITIONAL POSTERIORS

- $\mathbb{P}_i|Y^{(n)} \sim Dir(\alpha_{ij} + \sum_{m=1}^{n} 1_{Y_{m-1}=i,Y_m=j}; j \in E), \forall i \in E$

- $\lambda(i)|Y^{(n)}, X^{(n)} \sim G(a^*(i), b^*(i)), \forall i \in E$
  
  $a^*(i) = a(i) + \sum_{m=1}^{n} 1_{Y_m=i}$ & $b^*(i) = b(i) + \sum_{m=1}^{n} 1_{Y_m=i}X_m$

  $X^{(n)} = (X_1, \ldots, X_n)$

- $\pi(Y_m|Y^{(-m)}, \lambda(Y_m), X^{(n)}, \mathbb{P}) \propto P_{Y_{m-1},Y_m} \lambda(Y_m)e^{-\lambda(Y_m)X_m}P_{Y_m,Y_{m+1}}$

  $\sum_{j \in E} P_{Y_{m-1},j}\lambda(j)e^{-\lambda(j)X_m}P_{j,Y_{m+1}}$ normalizing constant

  $Y^{(-m)} = (Y_1, \ldots, Y_{m-1}, Y_{m+1}, \ldots, Y_n)$
POSTERIOR SAMPLE AND QUANTITIES

Gibbs sampling: posterior sample from $\pi(\Theta|X^{(n)})$ by iteratively drawing from the given full conditional posterior distributions

Posterior predictive distribution of $X_{n+1}$ after observing $X^{(n)}$

$$\pi(X_{n+1}|X^{(n)}) = \sum_{j \in E} \int \pi(X_{n+1}|\lambda(j))P_{Y_{n},j}\pi(\Theta|X^{(n)})d\Theta,$$

approximated as a Monte Carlo integral via

$$\pi(X_{n+1}|X^{(n)}) \approx \frac{1}{G} \sum_{g=1}^{G} \pi(X_{n+1}|\lambda^{g}(Y^{g}_{n+1}))$$

with $Y^{g}_{n+1}$ sampled, given the posterior sample $Y^{g}_{n}$, using the Dirichlet posterior on $\mathbb{P}_{Y^{g}_{n}}$
POSSIBLE EXTENSIONS

Selection of number of hidden states via Reversible Jump MCMC (Green, 1995) ⇒ allows for simulation of posterior distributions in parameter spaces of variable size

Ordered $\lambda(1) > \lambda(2) > \ldots > \lambda(k)$

RJMCMC with steps

- [Move] $\lambda_i$ changed to another value in $(\lambda_{i-1}, \lambda_{i+1})$

- [Death] Merge $\lambda_i$ and $\lambda_{i+1}$ into $\lambda_i^*$ and rearrange indices

- [Birth] Split $\lambda_i$ into $\lambda_{i,1}$ and $\lambda_{i,2}$ and rearrange indices
POSSIBLE EXTENSIONS

• Prior for $Y_0$

• Dynamic models for $\lambda$

• Nonhomogeneous Markov chain

• Estimation of stationary distribution (?)
JELINSKI-MORANDA DATA

34 software failure times

2 states for $Y_m$

$P_i \sim Beta(1,1), i = 1, 2$ (uniform)

$\lambda(i) \sim G(0.01, 0.01), i = 1, 2$ (diffuse)

5000 iterations

Convergence of Gibbs sampler pretty good
JELINSKI-MORANDA DATA

Posterior Distribution of Lambda[1]

Posterior Distribution of Lambda[2]
JELINSKI-MORANDA DATA

Posterior Distribution of $P[1,1]$

Posterior Distribution of $P[1,2]$

Posterior Distribution of $P[2,1]$

Posterior Distribution of $P[2,2]$
JELINSKI-MORANDA DATA

Posterior Predictive Density of $X_{35}$
**JELINSKI-MORANDA DATA**

Posterior Probabilities of State 1 over Time

| $m$ | $X_m$ | $P(Y_m = 1|D)$ | $m$ | $X_m$ | $P(Y_m = 1|D)$ | $m$ | $X_m$ | $P(Y_m = 1|D)$ |
|-----|-------|----------------|-----|-------|----------------|-----|-------|----------------|
| 1   | 9     | 0.8486         | 2   | 12    | 0.8846         | 3   | 11    | 0.9272         |
| 4   | 4     | 0.9740         | 5   | 7     | 0.9792         | 6   | 2     | 0.9874         |
| 7   | 5     | 0.9810         | 8   | 8     | 0.9706         | 9   | 5     | 0.9790         |
| 10  | 7     | 0.9790         | 11  | 1     | 0.9868         | 12  | 6     | 0.9812         |
| 13  | 1     | 0.9872         | 14  | 9     | 0.9696         | 15  | 4     | 0.9850         |
| 16  | 1     | 0.9900         | 17  | 3     | 0.9886         | 18  | 3     | 0.9858         |
| 19  | 6     | 0.9714         | 20  | 1     | 0.9584         | 21  | 11    | 0.7100         |
| 22  | 33    | 0.2036         | 23  | 7     | 0.3318         | 24  | 91    | 0.0018         |
| 25  | 2     | 0.6012         | 26  | 1     | 0.6104         | 27  | 87    | 0.0020         |
| 28  | 47    | 0.0202         | 29  | 12    | 0.2788         | 30  | 9     | 0.2994         |
| 31  | 135   | 0.0006         | 32  | 258   | 0.0002         | 33  | 16    | 0.1464         |
| 34  | 35    | 0.0794         |     |       |                |     |       |                |

*Expected posterior probability of the "bad" state decreases as we observe longer failure times*
MUSA SYSTEM 1 DATA

136 software failure times

2 states for $Y_m$

$\mathbb{P}_i \sim \text{Beta}(1, 1), i = 1, 2 \ (\text{uniform})$

$\lambda(i) \sim \mathcal{G}(0.01, 0.01), i = 1, 2 \ (\text{diffuse})$

5000 iterations

Convergence of Gibbs sampler pretty good
MUSA SYSTEM 1 DATA

Time Series Plot of Failure Times

Period
0 20 40 60 80 100 120 140
0 1000 2000 3000 4000 5000 6000
MUSA SYSTEM 1 DATA

Posterior Distribution of $\Lambda[1]$

Posterior Distribution of $\Lambda[2]$
MUSA SYSTEM 1 DATA

Posterior Distribution of $P_{1,1}$

Posterior Distribution of $P_{1,2}$

Posterior Distribution of $P_{2,1}$

Posterior Distribution of $P_{2,2}$
MUSA SYSTEM 1 DATA

Expected posterior probability of the "good" state increases as we observe longer failure times
SELF-EXCITING PROCESS WITH LATENT VARIABLES

NHPPs widely used in (software) reliability, characterised by an intensity function $\mu(t)$

Self-exciting processes (SEPs) add extra terms $g(t - t_i)$ to the intensity as a consequence of events at $t_i$ (e.g. introduction of new bugs)

Binary latent variables modelling the introduction of new bugs
⇒ SEP with intensity $\lambda(t) = \mu(t) + \sum_{j=1}^{N(t^{-})} Z_j g_j(t - t_j)$

- $\mu(t)$ intensity of process w/o introduction of new bugs
- $N(t^{-})$ number of failures right before $t$
- $t_1 < t_2 < \ldots < t_n$ failures in $(0, T]$ for $u > 0$ and $= 0$ o.w.
SELF-EXCITING PROCESS: LIKELIHOOD

\( t = (t_1, \ldots, t_n) \) failures in \((0, T]\)

\( Z = (Z_1, \ldots, Z_n) \) latent variables at \( t = (t_1, \ldots, t_n) \)

Likelihood \( L(\theta; t, Z) = f(t|Z, \theta)f(Z|\theta) \)

\[
f(t|Z, \theta) = \prod_{i=1}^{n} \lambda(t_i) e^{-\int_0^T \lambda(t) dt}
\]

\[
= \prod_{i=1}^{n} \left[ \mu(t_i) + \sum_{j=1}^{i-1} Z_j g(t_i - t_j) \right] e^{-\int_0^T \mu(t) dt - \sum_{j=1}^{N(T^-)} Z_j \int_0^{T-t_j} g_j(t) dt}
\]

[\( \theta \) omitted]
SELF-EXCITING PROCESS: ASSUMPTIONS

PLP $\mu(t) = M \beta t^{\beta - 1}$, $M > 0, \beta > 0$ & $\mu \equiv g_j, \forall j$

Other possibilities (still to be explored)

- $(M_i, \beta_i) \neq (M_j, \beta_j), i \neq j$ & $(M_j, \beta_j) \neq (M_0, \beta_0) [i.e. \mu(t)], \forall j$

- $M_0 > M_1 > \ldots > M_n,, e.g. M_j = M_0 \gamma^j, 0 < \gamma < 1$

- $M_j = \alpha M_{j-1} + \delta, 0 < \alpha < 1, \delta > -\alpha M_{j-1}$

- $\mu(t) = M \beta e^{-\beta t}$, with $\int_0^\infty \mu(dt) = M < \infty$
SELF-EXCITING PROCESS: LIKELIHOOD

\[
f(t|Z, \theta) = M^n \beta^n \prod_{i=1}^{n} \left[ t_i^{\beta-1} + \sum_{j=1}^{i-1} Z_j (t_i - t_j) \right] e^{-M \left[ T^{\beta} + \sum_{j=1}^{N(T^-)} Z_j (T - t_j)^{\beta} \right]}
\]

\[
= M^n \beta^n \prod_{i=1}^{n} A_i(\beta, Z^{(i-1)}) e^{-MB(\beta, Z^{(n)})}
\]

- \( Z^{(i)} = (Z_1, \ldots, Z_i) \)
- \( A_i(\beta, Z^{(i-1)}) = t_i^{\beta-1} + \sum_{j=1}^{i-1} Z_j (t_i - t_j) \)
- \( B(\beta, Z^{(n)}) = T^{\beta} + \sum_{j=1}^{N(T^-)} Z_j (T - t_j)^{\beta} \)
SELF-EXCITING PROCESS: LIKELIHOOD

\[ Z_j \sim Bern(p_j), \forall j \]

\[
f(t, Z|\theta) = f(t|Z, \theta)f(Z|\theta) = f(t|Z, \theta) \prod_{i=1}^{n} p_j^{Z_j}(1-p_j)^{1-Z_j}
\]

Two possibilities

- Sum over all \( Z^{(n)} \Rightarrow f(t|\theta) \)

- Treat \( Z_j \)'s as parameters and look for full conditionals (for MCMC) \( [\text{we follow this}] \)
SELF-EXCITING PROCESS: PRIORS

- $M \sim \mathcal{G}(\alpha, \delta)$
- $\beta \sim \mathcal{G}(\rho, \lambda)$
- $p_j \sim \text{Beta}(\mu_j, \sigma_j), \forall j$

Other possibilities (still to be explored)
- $p_j = \phi p_{j-1} + \eta$ (provided $0 \leq p_j \leq 1$)
- (A more general) Markov chain for $p_j$’s
- $p_j \sim \text{Beta}(\mu, \sigma), \forall j$
SELF-EXCITING PROCESS: NOTATIONS

\[ p = (p_1, \ldots, p_n) \]

\[ p_{-j} = (p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n) \]

\[ Z_{-j} = (Z_1, \ldots, Z_{j-1}, Z_{j+1}, \ldots, Z_n) \]

From now on, omit dependence on \( t = (t_1, \ldots, t_n) \)
SELF-EXCITING PROCESS: POSTERIORS

- \( M|\beta, Z^{(n)}, p \sim G(\alpha + n, \delta + B(\beta, Z^{(n)})) \)

- \( \beta|M, Z^{(n)}, p \propto \beta^{\rho+n} \prod_{i=1}^{n} A_i(\beta, Z^{(i-1)}) e^{-MB(\beta, Z^{(n)})-\lambda\beta} \)

- \( p_j|M, \beta, Z^{(n)}, p_{-j} \sim Beta(\mu_j + Z_j, \sigma_j + (1 - Z_j)), \forall j \)
SELF-EXCITING PROCESS: POSTERIORS

\[ P(Z_j = r|M, \beta, p, Z_{-j}) = \frac{C_r}{C_0 + C_1}, r = 0, 1 \]

\[ C_0 = \prod_{i=j+1}^{n} \left[ t_i^{\beta - 1} + \sum_{h=1, i-1; h \neq j} Z_h (t_i - t_h)^\beta \right] \]

\[ C_1 = \prod_{i=j+1}^{n} \left[ t_i^{\beta - 1} + \sum_{h=1, i-1; h \neq j} Z_h (t_i - t_h)^\beta + (t_i - t_j)^\beta \right] e^{-M(T-t_j)^\beta} \]
SELF-EXCITING PROCESS: EXTENSIONS

Different \((M_i, \beta_i)\) ⇒ messy computations in general

\[ M_i = M \rho^i, \beta_i = \beta, 0 < \rho < 1, i = 1, \ldots, n \]

\[ \rho \sim Beta(\phi, \tau) \]

- Full conditional posterior for \(\rho\), apart from constant

- Same full conditionals for other parameters with changes:
  \((t_i - t_j) \rightarrow \rho^j(t_i - t_j) \& (T - t_j) \rightarrow \rho^j(T - t_j)\)
CHANGE POINTS

F.R. and Sivaganesan (2005)

NHPP with $\lambda(t; M, \beta) = Mg(t, \beta), M, \beta > 0$

- Changes at each failure time
- Changes at a random number of failure times
- Changes at a random number of times
CHANGE POINTS

Changes at each failure time

- Hierarchical Model
  - $\beta_i \sim \text{i.i.d. } LN(\phi, \sigma^2), i = 0, \ldots, n$
  - $\phi \sim N(\mu, \tau^2)$
  - $\sigma^2 \sim IG(\rho, \gamma)$

- Gamma prior for $M$

- Conditional posteriors
  - Gamma for $M$
  - Inverse Gamma for $\sigma^2$
  - Normal for $\phi$
  - Known (apart from a constant) for $\beta_i$'s

$\Rightarrow$ Metropolis-Hastings and Gibbs sampling
CHANGE POINTS

Changes at each failure time

- **Dynamic Model**
  - \( \log \beta_i = \log a + \log \beta_{i-1} + \epsilon_i, i = 1, \ldots, n \)
  - \( \epsilon_i \sim N(0, \sigma^2) \)

- **Priors**
  - Gamma for \( M \)
  - Inverse Gamma for \( \sigma^2 \)
  - Lognormal for \( a \mid \sigma^2 \)
  - Lognormal for \( \beta_0 \mid \sigma^2 \)
CHANGE POINTS

Changes at each failure time

- Conditional posteriors
  - Gamma for $M$
  - Inverse Gamma for $\sigma^2$
  - Lognormal for $a$
  - Known (apart from a constant) for $\beta_i$’s
  $\implies$ Metropolis-Hastings and Gibbs sampling

- Extension $\Rightarrow$ dynamic linear model ($i = 1, \ldots, n$)
  $\log \beta_i = a \theta_i + \epsilon_i$
  $\theta_i = b \theta_{i-1} + \delta_i$
CHANGE POINTS

Changes at a random number of failures

- Dynamic model as before
- Bernoulli r.v.’s for change/no change
- Beta priors on Bernoulli parameter

Changes at a random number of points

⇒ Reversible jump MCMC with steps:

- change of $M$ and $\beta$ at a randomly chosen change point
- change to the location of a randomly chosen change point
- “birth” of a new change point at a randomly chosen location in $(0, y]$;
- “death” of a randomly chosen change point
FUTURE RESEARCH

• Optimal testing policy

• Extensive data analysis (with Tom Mazzuchi)