Solving Coalitional Games with Cimmino Algorithm

Tomáš Kroupa

Department of Decision-Making Theory
Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic

in collaboration with:

Dan Butnariu
Department of Mathematics
University of Haifa
Israel

Coalitional Games

(von Neumann, Morgenstern; 1953)

- models of interacting decision-makers that focus on the behavior of groups of players
- every coalition acts in the name of its members to realize its potential worth and distribute it among its members

Coalitional Games

(von Neumann, Morgenstern; 1953)

- models of interacting decision-makers that focus on the behavior of groups of players
- every coalition acts in the name of its members to realize its potential worth and distribute it among its members

A coalitional game consists of

1. a set of players
2. a set of coalitions
3. a potential worth of every coalition

- a solution of a coalitional game can be determined by a set of payoffs accepted by all coalitions and players

Games with Fuzzy Coalitions

(J.-P. Aubin, 1974)

$N = \{1, \ldots, n\}$	set of players
$a = (a_1, \ldots, a_n) \in [0,1]^n$	(fuzzy) coalition
$a \in \{0,1\}^n$	crisp coalition
$0 = (0, \ldots, 0)$	empty coalition
$1 = (1, \ldots, 1)$	grand coalition

A crisp coalition $a \in \{0,1\}^n$ can be identified with a subset of N.

Games with Fuzzy Coalitions

(J.-P. Aubin, 1974)

$N = \{1, \dots, n\}$	set of players
$a = (a_1, \dots, a_n) \in [0, 1]^n$	(fuzzy) coalition
$a \in \{0, 1\}^n$	crisp coalition
$0 = (0, \dots, 0)$	empty coalition
$1 = (1, \dots, 1)$	grand coalition

A crisp coalition $a \in \{0, 1\}^n$ can be identified with a subset of N.

Definition

A game (with fuzzy coalitions) is a function

$$v : [0, 1]^n \to \mathbb{R} \quad \text{with } v(0) = 0.$$

A game (with crisp coalitions) is a set function

$$v : \{0, 1\}^n \to \mathbb{R} \quad \text{with } v(0) = 0.$$

From Coalitions to Fuzzy Coalitions

- $[0,1]^n = \text{co } \{0,1\}^n$
- every player $i \in N$ conforms with a status of a fuzzy coalition a in a certain degree $a_i \in [0,1]$
- most set-theoretic notions can be generalized to $[0,1]^n$
- real functions of n variables can be processed more easily than combinatorially difficult set functions on finite sets
- forming a multilinear extension $v : [0,1]^n \to \mathbb{R}$ of a game with crisp coalitions $v_0 : \{0,1\}^n \to \mathbb{R}$:

$$v(a) = \sum_{b \in \{0,1\}^n \setminus \{0\}} \left(\prod_{i | b_i = 1} a_i \prod_{j | b_j = 0} (1 - a_j) \right) v_0(b)$$

Superadditive Games

- the idea of l'union fait la force

Definition

A game v is called superadditive when

$$a, b \in [0,1]^n,\ a + b \leq 1 \quad \Rightarrow \quad v(a) + v(b) \leq v(a+b)$$

- the trend to coalesce is extreme in superadditive games...
- ...the players have an incentive to form the grand coalition 1

Solution Concept: Core

$x \in \mathbb{R}^n$	vector of individual payoffs
$\langle a, x \rangle$	payoff of the coalition a

Solution Concept: Core

$x \in \mathbb{R}^n$	vector of individual payoffs
$\langle a, x \rangle$	payoff of the coalition a

Definition (Aubin; 1974)

Let v be a game. The core of v is the set

$$\mathbf{C}(v) = \{x \in \mathbb{R}^n \mid \langle 1, x \rangle = v(1) \text{ and } \langle a, x \rangle \geq v(a), \forall a \in [0,1]^n \backslash \{1\}\}$$

Put

$$C_a(v) = \begin{cases} \{x \in \mathbb{R}^n \mid \langle a, x \rangle \geq v(a)\}, & \text{if } a \in [0,1]^n \setminus \{1\}, \\ \{x \in \mathbb{R}^n \mid \langle 1, x \rangle = v(1)\}, & \text{if } a = 1. \end{cases}$$

The core $\mathbf{C}(v)$ is a closed convex set in $\mathbb{R}^n$ and $\mathbf{C}(v) = \bigcap_{a \in [0,1]^n} C_a(v)$

Examples of Cores

$N = \{1, 2\}$

Example (empty core)

$$u(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \leq 1, \\ 1, & \text{otherwise.} \end{cases}$$

$\mathbf{C}(u) = \emptyset$ since the hyperplane $x_1 + x_2 = 1$ misses the halfspace $\frac{2}{3}x_1 + \frac{2}{3}x_2 \geq 1$.

Examples of Cores

$N = \{1, 2\}$

Example (empty core)

$$u(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \leq 1, \\ 1, & \text{otherwise.} \end{cases}$$

$\mathbf{C}(u) = \emptyset$ since the hyperplane $x_1 + x_2 = 1$ misses the halfspace $\frac{2}{3}x_1 + \frac{2}{3}x_2 \geq 1$.

Example (polyhedral core)

$$w(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \leq 1, \\ a_1 + a_2 - 1, & \text{otherwise.} \end{cases}$$

$$\mathbf{C}(w) = \bigcap_{a \in \{0,1\}^2} C_a(w) = \{x \in [0,1]^2 \mid x_1 + x_2 = 1\}$$

Examples of Cores (ctnd.)

$N = \{1, \ldots, n\}$

Example

$(f_j)_{j \in J}$ ... family of concave and PH functions $\mathbb{R}^n \to \mathbb{R}$

$$\boxed{v(a) = \inf \{f_j(a) \mid j \in J\}, \quad \forall a \in \mathbb{R}^n}$$

The game $v \restriction [0,1]^n$ is PH, superadditive, and

$$\mathbf{C}(v) = \{x \in \mathbb{R}^n \mid \langle 1 - c, x \rangle \leq v(1) - v(c), \forall c \in \mathbb{R}^n\} \neq \emptyset$$

Characterizations of Core

Theorem (Aubin, 1981)

Let v be a PH game. If v is superadditive, then $\boxed{\mathbf{C}(v) = \partial v(1)}$.

If v is both superadditive and continuously differentiable at 1, then $\boxed{\mathbf{C}(v) = \{\nabla v(1)\}}$.

Characterizations of Core

Theorem (Aubin, 1981)

Let v be a PH game. If v is superadditive, then $\boxed{\mathbf{C}(v) = \partial v(1)}$.
If v is both superadditive and continuously differentiable at 1, then $\boxed{\mathbf{C}(v) = \{\nabla v(1)\}}$.

Theorem (Tijs et al., 2003)

Let v be a game such that for every $a, b, d, b + d \in [0, 1]^n$:

$$a \leq b \quad \Rightarrow \quad v(a + d) - v(a) \leq v(b + d) - v(b)$$

Then $\mathbf{C}(v) \neq \emptyset$ *and* $\boxed{\mathbf{C}(v) = \bigcap_{a \in \{0,1\}^n} C_a(v)}$

Characterizations of Core

Theorem (Aubin, 1981)

Let v be a PH game. If v is superadditive, then $\boxed{\mathbf{C}(v) = \partial v(1)}$.
If v is both superadditive and continuously differentiable at 1, then $\boxed{\mathbf{C}(v) = \{\nabla v(1)\}}$.

Theorem (Tijs et al., 2003)

Let v be a game such that for every $a, b, d, b + d \in [0, 1]^n$:

$$a \leq b \quad \Rightarrow \quad v(a + d) - v(a) \leq v(b + d) - v(b)$$

$$\text{Then } \mathbf{C}(v) \neq \emptyset \text{ and } \boxed{\mathbf{C}(v) = \bigcap_{a \in \{0,1\}^n} C_a(v)}$$

Theorem (Azrieli, Lehrer; 2007)

A game v has a nonempty core $\mathbf{C}(v)$ iff

$$v(1) = \sup \left\{ \sum_{i=1}^{m} \lambda_i v(a^i) \mid m \in \mathbb{N}, 1 = \sum_{i=1}^{m} \lambda_i a^i, \lambda_j \geq 0, a^j \in [0, 1]^n \right\}$$

Core Difficulties

- checking nonemptiness of $\mathbf{C}(v)$ is hard...
- ...even when the core $\mathbf{C}(v)$ is polyhedral:

$$n = 20 \quad \Rightarrow \quad \bigcap_{a \in \{0,1\}^{20}} C_a(v)$$

is given by more than 10^6 affine constraints

- core concept assumes that the game is a one-shot affair: all fuzzy coalitions come up with their demands simultaneously
- the solution can be blocked by a group of fuzzy coalitions whose role in negotiations is "negligible"

Enlarged Core

$\mathfrak{A}$	Lebesgue measurable subsets of $[0,1]^n$
μ	complete probability measure on $\mathfrak{A}$

$\mu(A)$ is a measure of influence of the fuzzy coalitions in A $\Rightarrow$ every $A \in \mathfrak{A}$ with $\mu(A) = 0$ is viewed as "negligible"

Definition

The set

$$\mathbf{C}_\mu(v) = \bigcup_{\substack{A \in \mathfrak{A}: \\ \mu(A)=0}} \bigcap_{a \in [0,1]^n \setminus A} C_a(v)$$

is called the enlarged core of v w.r.t. μ.

Always

$$\mathbf{C}(v) \subseteq \mathbf{C}_\mu(v).$$

Characterization of Enlarged Core

$A_x = \{a \in [0,1]^n \mid x \in C_a(v)\}$	coalitions accepting the payoff x
$\{x \in \mathbb{R}^n \mid \mu(A_x) = 1\}$	μ-almost common points of $C_a(v)$

Theorem

Let v be a game and μ be a complete probability measure on $\mathfrak{A}$. If the function v is Lebesgue measurable, then $A_x \in \mathfrak{A}$ for every $x \in \mathbb{R}^n$, and

$$\mathbf{C}_\mu(v) = \{x \in \mathbb{R}^n \mid \mu(A_x) = 1\}.$$

- the enlarged core $\mathbf{C}_\mu(v)$ thus coincides with the set of payoffs accepted by "large" sets of coalitions
- the set $\mathbf{C}_\mu(v)$ can be much greater than $\mathbf{C}(v)$...

Enlarged Core Can Be Too Large...

Example

$$w(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \leq 1, \\ a_1 + a_2 - 1, & \text{otherwise.} \end{cases}$$

$\mathbf{C}(w) = \{x \in [0,1]^2 \mid x_1 + x_2 = 1\}$

If $\mu = \lambda$, then $\lambda(\{1\}) = 0$ and

$$\alpha_1 e_1 + \alpha_2 e_2 \in \mathbf{C}_\mu(w) \quad \text{for every } \alpha_1, \alpha_2 \geq 0$$

Core vs. Enlarged Core

Theorem

Let v be a continuous game and μ be a complete probability measure on $\mathfrak{A}$ such that, for every $A \in \mathfrak{A}$, $\mu(A) > 0$ whenever A is open or $1 \in A$. Then

$$\boxed{\mathbf{C}(v) = \mathbf{C}_{\mu}(v)}$$

Core vs. Enlarged Core

Theorem

Let v be a continuous game and μ be a complete probability measure on $\mathfrak{A}$ such that, for every $A \in \mathfrak{A}$, $\mu(A) > 0$ whenever A is open or $1 \in A$. Then

$$\boxed{\mathbf{C}(v) = \mathbf{C}_\mu(v)}$$

Example (1)

$\lambda \ldots$ Lebesgue measure, $\delta_1 \ldots$ Dirac measure concentrated at 1

$$\mu_1 = \alpha\lambda + (1-\alpha)\delta_1, \quad \alpha \in (0,1)$$

Core vs. Enlarged Core

Theorem

Let v be a continuous game and μ be a complete probability measure on $\mathfrak{A}$ such that, for every $A \in \mathfrak{A}$, $\mu(A) > 0$ whenever A is open or $1 \in A$. Then

$$\boxed{\mathbf{C}(v) = \mathbf{C}_\mu(v)}$$

Example (1)

$\lambda \ldots$ Lebesgue measure, $\delta_1 \ldots$ Dirac measure concentrated at 1

$$\mu_1 = \alpha\lambda + (1-\alpha)\delta_1, \quad \alpha \in (0,1)$$

Example (2)

Let $v > 0$ λ-a.e. and ν be given by the density $\frac{v}{\int v d\lambda}$

$$\mu_2 = \alpha\nu + (1-\alpha)\delta_1, \quad \alpha \in (0,1)$$

Bargaining Schemes

- instead of describing the whole (enlarged) core, try to recover at least one point in it:

Definition

A bargaining scheme for the (enlarged) core of a game v is an iterative procedure, which starts from an initial payoff $x^0 \in \mathbb{R}^n$ and generates a sequence $(x^k)_{k \in \mathbb{N}}$ in $\mathbb{R}^n$ converging to a point of the (enlarged) core (provided such a point exists).

Bargaining Schemes

- instead of describing the whole (enlarged) core, try to recover at least one point in it:

Definition

A bargaining scheme for the (enlarged) core of a game v is an iterative procedure, which starts from an initial payoff $x^0 \in \mathbb{R}^n$ and generates a sequence $(x^k)_{k \in \mathbb{N}}$ in $\mathbb{R}^n$ converging to a point of the (enlarged) core (provided such a point exists).

- games with crisp coalitions: coalitions arrive at a payoff from core in cycles (Wu; 1977)
- games with fuzzy coalitions: iterative projections method for solving general convex feasibility problems can be used

Convex Feasibility Problems

(Bauschke, Borwein; 1996)
Given k closed convex subsets $C_1, \ldots, C_k$ in a Hilbert space with a nonempty intersection

$$C = \bigcap_{i=1}^{k} C_i,$$

find some point $x \in C$.

Convex Feasibility Problems

(Bauschke, Borwein; 1996)
Given k closed convex subsets $C_1, \ldots, C_k$ in a Hilbert space with a nonempty intersection

$$C = \bigcap_{i=1}^{k} C_i,$$

find some point $x \in C$.

(von Neumann; 1953) the "cyclic" projection method

Cimmino Type Bargaining Scheme: Assumptions

Let

- v be a Lebesgue measurable game
- μ be a complete probability measure on $\mathfrak{A}$
- the mapping $[0,1]^n \to \mathbb{R}$ defined as

$$a \in [0,1]^n \mapsto \begin{cases} v^2(a)/\|a\|^2, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0, \end{cases}$$

  is μ-integrable

Cimmino Type Bargaining Scheme

The projection $P_a x$ of x onto $C_a(v)$:

$$P_a x = \begin{cases} x + \frac{\max\{0, v(a) - \langle a, x \rangle\}}{\|a\|^2} a, & \text{if } a \in [0,1]^n \setminus \{0, 1\}, \\ x + \frac{v(1) - \langle 1, x \rangle}{n} 1, & \text{if } a = 1, \\ x, & \text{if } a = 0. \end{cases}$$

The "amalgamated" projection $\mathbf{P} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\boxed{\mathbf{P}x = \int_{[0,1]^n} (P_a x) \, d\mu(a), \quad \forall x \in \mathbb{R}^n}$$

Cimmino Type Bargaining Scheme

The projection $P_a x$ of x onto $C_a(v)$:

$$P_a x = \begin{cases} x + \frac{\max\{0, v(a) - \langle a, x \rangle\}}{\|a\|^2} a, & \text{if } a \in [0,1]^n \setminus \{0, 1\}, \\ x + \frac{v(1) - \langle 1, x \rangle}{n} 1, & \text{if } a = 1, \\ x, & \text{if } a = 0. \end{cases}$$

The "amalgamated" projection $\mathbf{P} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\boxed{\mathbf{P}x = \int_{[0,1]^n} (P_a x)\, d\mu(a), \quad \forall x \in \mathbb{R}^n}$$

Definition

The Cimmino type bargaining scheme in the game v is the following rule of generating sequences $(x^k)_{k \in \mathbb{N}}$ in $\mathbb{R}^n$:

$$\boxed{x^0 \in \mathbb{R}^n \quad \text{and} \quad x^{k+1} = \mathbf{P}x^k, \ \forall k \in \mathbb{N}_0}$$

Cimmino Type Bargaining Scheme (contd.)

Define

$$\mathbf{g}(x) = \frac{1}{2} \int_{[0,1]^n} \|P_a x - x\|^2 \, d\mu(a), \quad x \in \mathbb{R}^n.$$

Theorem

The mapping $\mathbf{g}$ *is*

- *nonnegative and everywhere finite*
- *convex*
- *continuously differentiable with* $\nabla \mathbf{g}(x) = x - \mathbf{P}x$

Cimmino Type Bargaining Scheme (contd.)

Theorem (Recovering a point in the enlarged core)

Let $(x^k)_{k\in\mathbb{N}}$ be the sequence generated by the Cimmino type bargaining scheme with an initial point $x^0 \in \mathbb{R}^n$.

1. *If $(x^k)_{k\in\mathbb{N}}$ is bounded, then*

Cimmino Type Bargaining Scheme (contd.)

Theorem (Recovering a point in the enlarged core)

Let $(x^k)_{k\in\mathbb{N}}$ be the sequence generated by the Cimmino type bargaining scheme with an initial point $x^0 \in \mathbb{R}^n$.

1. *If $(x^k)_{k\in\mathbb{N}}$ is bounded, then*
   - *for every $y^0 \in \mathbb{R}^n$, the sequence $(y^k)_{k\in\mathbb{N}}$ generated by the Cimmino type bargaining scheme is also bounded*

Cimmino Type Bargaining Scheme (contd.)

Theorem (Recovering a point in the enlarged core)

Let $(x^k)_{k\in\mathbb{N}}$ be the sequence generated by the Cimmino type bargaining scheme with an initial point $x^0 \in \mathbb{R}^n$.

1. *If $(x^k)_{k\in\mathbb{N}}$ is bounded, then*
   - *for every $y^0 \in \mathbb{R}^n$, the sequence $(y^k)_{k\in\mathbb{N}}$ generated by the Cimmino type bargaining scheme is also bounded*
   - *the limit $x^* = \lim_{k\to\infty} x^k$ exists,*

     x^* is a minimizer of $\mathbf{g}$ and $\mathbf{g}(x^*) = \lim_{k\to\infty} \mathbf{g}(x^k)$.

     Moreover, if $\mathbf{g}(x^*) = 0$, then $\boxed{x^* \in \mathbf{C}_\mu(v)}$

Cimmino Type Bargaining Scheme (contd.)

Theorem (Recovering a point in the enlarged core)

Let $(x^k)_{k\in\mathbb{N}}$ be the sequence generated by the Cimmino type bargaining scheme with an initial point $x^0 \in \mathbb{R}^n$.

1. *If $(x^k)_{k\in\mathbb{N}}$ is bounded, then*
   - *for every $y^0 \in \mathbb{R}^n$, the sequence $(y^k)_{k\in\mathbb{N}}$ generated by the Cimmino type bargaining scheme is also bounded*
   - *the limit $x^* = \lim_{k\to\infty} x^k$ exists,*

     x^* is a minimizer of $\mathbf{g}$ and $\mathbf{g}(x^*) = \lim_{k\to\infty} \mathbf{g}(x^k)$.

     Moreover, if $\mathbf{g}(x^*) = 0$, then $\boxed{x^* \in \mathbf{C}_\mu(v)}$
2. *If $(x^k)_{k\in\mathbb{N}}$ is unbounded or $\mathbf{g}(x^*) \neq 0$, then $\boxed{\mathbf{C}_\mu(v) = \emptyset}$ and (a fortiori) $\mathbf{C}(v) = \emptyset$.*

Cimmino Type Bargaining Scheme (contd.)

Theorem (Recovering a point in the core)

Let v be a continuous game and μ be a complete probability measure on $\mathfrak{A}$ such that, for every $A \in \mathfrak{A}$, $\mu(A) > 0$ whenever A is open or $1 \in A$. If $(x^k)_{k \in \mathbb{N}}$ is a bounded sequence generated by the Cimmino type bargaining scheme s.t. $\mathbf{g}(x^*) = 0$, then

$$\boxed{x^* \in \mathbf{C}(v)}$$

Experiments

$$w(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \leq 1, \\ a_1 + a_2 - 1, & \text{otherwise.} \end{cases}$$

$$\mathbf{C}(w) = \{x \in [0,1]^2 \mid x_1 + x_2 = 1\}$$

$$\boxed{\mu = \alpha\lambda + (1-\alpha)\delta_1}$$

Experiments

$$w(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \le 1, \\ a_1 + a_2 - 1, & \text{otherwise.} \end{cases}$$

$\mathbf{C}(w) = \{x \in [0,1]^2 \mid x_1 + x_2 = 1\}$

$$\boxed{\mu = \alpha\lambda + (1-\alpha)\delta_1}$$

1. $x^0 = (0,0), \alpha = 0.9 \quad \Rightarrow$
   54 steps, $x^* = (0.499277, 0.499277)$, $\mathbf{g}(x^*) = 3.64665 \times 10^{-13}$

Experiments

$$w(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \leq 1, \\ a_1 + a_2 - 1, & \text{otherwise.} \end{cases}$$

$\mathbf{C}(w) = \{x \in [0,1]^2 \mid x_1 + x_2 = 1\}$

$$\boxed{\mu = \alpha\lambda + (1-\alpha)\delta_1}$$

1. $x^0 = (0,0), \alpha = 0.9 \quad \Rightarrow$
   54 steps, $x^* = (0.499277, 0.499277)$, $\mathbf{g}(x^*) = 3.64665 \times 10^{-13}$
2. $x^0 = (0,0)$, $\alpha = 0.5 \quad \Rightarrow$
   9 steps, $x^* = (0.499297, 0.499297)$, $\mathbf{g}(x^*) = 3.24784 \times 10^{-13}$

Experiments

$$w(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \le 1, \\ a_1 + a_2 - 1, & \text{otherwise.} \end{cases}$$

$\mathbf{C}(w) = \{x \in [0,1]^2 \mid x_1 + x_2 = 1\}$

$\mu = \alpha\lambda + (1-\alpha)\delta_1$

1. $x^0 = (0,0), \alpha = 0.9 \quad \Rightarrow$
   54 steps, $x^* = (0.499277, 0.499277)$, $\mathbf{g}(x^*) = 3.64665 \times 10^{-13}$
2. $x^0 = (0,0)$, $\alpha = 0.5 \quad \Rightarrow$
   9 steps, $x^* = (0.499297, 0.499297)$, $\mathbf{g}(x^*) = 3.24784 \times 10^{-13}$
3. $x^0 = (50,70), \alpha = 0.9 \quad \Rightarrow$
   600 steps, $x^* = (-0.01843, 1.01977)$, $\mathbf{g}(x^*) = 7.31311 \times 10^{-7}$

Experiments

$$w(a_1, a_2) = \begin{cases} 0, & a_1 + a_2 \leq 1, \\ a_1 + a_2 - 1, & \text{otherwise.} \end{cases}$$

$\mathbf{C}(w) = \{x \in [0,1]^2 \mid x_1 + x_2 = 1\}$

$$\boxed{\mu = \alpha\lambda + (1-\alpha)\delta_1}$$

1. $x^0 = (0,0), \alpha = 0.9 \quad \Rightarrow$
   54 steps, $x^* = (0.499277, 0.499277)$, $\mathbf{g}(x^*) = 3.64665 \times 10^{-13}$
2. $x^0 = (0,0)$, $\alpha = 0.5 \quad \Rightarrow$
   9 steps, $x^* = (0.499297, 0.499297)$, $\mathbf{g}(x^*) = 3.24784 \times 10^{-13}$
3. $x^0 = (50,70), \alpha = 0.9 \quad \Rightarrow$
   600 steps, $x^* = (-0.01843, 1.01977)$, $\mathbf{g}(x^*) = 7.31311 \times 10^{-7}$
4. $x^0 = (50,70)$, $\alpha = 0.1 \quad \Rightarrow$
   600 steps, $x^* = (-0.31587, 1.31788)$, $\mathbf{g}(x^*) = 2.54637 \times 10^{-3}$

Further Directions

- the iterative computations of $\mathbf{P}$ require numerical integration
  $\Rightarrow$
  convergence behavior of Cimmino algorithm under numerical integration techniques is not known

Further Directions

- the iterative computations of **P** require numerical integration
  $\Rightarrow$
  convergence behavior of Cimmino algorithm under numerical integration techniques is not known
- accelerating the convergence via relaxed Cimmino algorithm:

$$x^0 \in \mathbb{R}^n \quad \text{and} \quad x^{k+1} = \alpha_k x^k + (1 - \alpha_k)\mathbf{P}_k x^k, \ \forall k \in \mathbb{N}_0$$

  where $(\alpha_k)_{k\in\mathbb{N}} \in (0,1]^{\mathbb{N}}$ and $(\mu_k)_{k\in\mathbb{N}}$ is a sequence of complete probability measures on $\mathfrak{A}$ $\Rightarrow$
  which is the proper choice of $(\alpha_k)_{k\in\mathbb{N}}$ and $(\mu_k)_{k\in\mathbb{N}}$?

Important Open Problem

Find a necessary and a sufficient condition
for the core $\mathbf{C}(v)$ to be a stable set of imputations!

Important Open Problem

Find a necessary and a sufficient condition
for the core $\mathbf{C}(v)$ to be a stable set of imputations!

An imputation in a game v is a vector $x \in \mathbb{R}^n$ s.t. $\langle 1, x \rangle = v(1)$ and $x_i \geq v(\{i\})$ for every $i \in N$. Further, $y \preceq x$ means that there exists $a \in [0,1]^n$ such that $\langle a, x \rangle \leq v(a)$ and $y_i < x_i$ for every $i \in N$ with $a_i > 0$.

Definition

A set $X \subseteq \mathbb{R}^n$ of imputations is called stable when

1. for every $x, y \in X$ it is not true that $x \preceq y$
2. for every imputation $y \notin X$ there is $x \in X$ with $y \preceq x$

Important Open Problem

Find a necessary and a sufficient condition
for the core $\mathbf{C}(v)$ to be a stable set of imputations!

An imputation in a game v is a vector $x \in \mathbb{R}^n$ s.t. $\langle 1, x \rangle = v(1)$ and $x_i \geq v(\{i\})$ for every $i \in N$. Further, $y \preceq x$ means that there exists $a \in [0,1]^n$ such that $\langle a, x \rangle \leq v(a)$ and $y_i < x_i$ for every $i \in N$ with $a_i > 0$.

Definition

A set $X \subseteq \mathbb{R}^n$ of imputations is called stable when

1. for every $x, y \in X$ it is not true that $x \preceq y$
2. for every imputation $y \notin X$ there is $x \in X$ with $y \preceq x$

References

J.-P. Aubin
Optima and Equilibria. An introduction to nonlinear analysis.
Springer-Verlag, 1998

D. Butnariu, B.Z. Shklyar
Existence and approximation of solutions for Fredholm equations of the first kind with applications to a linear moment problem.
Optimization Methods and Software, to appear.

H.H. Bauschke, J.M. Borwein
On projection algorithms for solving convex feasibility problems.
SIAM Review 38, 1996