Portfolio Selection Problems

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Outline

Diversification

[Markowitz 50's and 60's - Nobel Prize 1990]

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Markowitz model:

$$\boxed{\max_{\mathbf{1}'\pi=1,\pi\geq 0} \mathbb{E}(r'\pi) - \lambda \mathrm{var}(r'\pi)}$$

$r \in \mathbb{R}^N$ - stochastic asset returns
(i.e. a quadratic programming problem).

More generally:

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Theorem (von Neumann, Morgenstern)

Given some (reasonable) requirements, $\preceq$ may be represented by an (utility) function u, i.e.

$$x \preceq y \Leftrightarrow \mathbb{E}u(x) \leq \mathbb{E}u(y)$$

for all pairs of r.v.'s x, y.

Single stage problem

$$\text{(UM)} \qquad \boxed{\max_{\mathbf{1}'\pi=1} \mathbb{E}u(r'\pi),}$$

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Too general to solve.

On the other hand, the solution of

$$\text{(MV)} \qquad \boxed{\min_{\mathbf{1}'\pi=1, \mathbb{E}(r'\pi)=\mu} \operatorname{var}(r'\pi)}$$

is closed form and unique for each $\mu \in \mathbb{R}$ given that $\operatorname{var}(r)$ is regular.

Capital asset pricing model

[Sharpe, 60's - Nobel Prize 1990]

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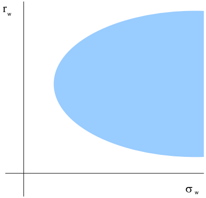

All portfolios of risky assets in the mean / st. dev. plane

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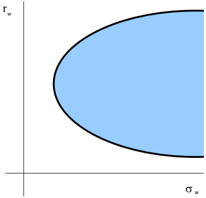
Solutions of [MV]



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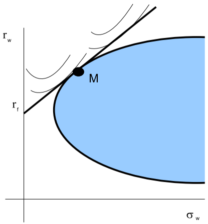

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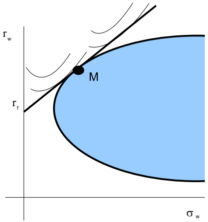

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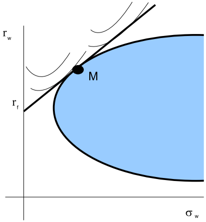

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- After the emergence of CAPM, mutual funds “holding index” appeared. $\Rightarrow$ After inclusion of a stock into an index (no economic significance) its price jumps up which is explained by the activity of those funds.
- CAPM remains valid even without the possibility of short sales, there exist various modifications.

Multi-period models

[Samuelson 1969 (Nobel Prize 1970), Hakansson 1970]

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$$\max_{c_0,\dots,c_T,\pi_0,\dots,\pi_T} \mathbb{E}\left[\sum_{t=0}^{T} \rho^{-i} u_1(c_i) + u_2(w_T)\right]$$

$$w_t = \mathbf{1}'\pi_t + c_t, \quad w_{t+1} = r_t'\pi_t, \quad c_t, w_t \geq 0,$$
$$c_t, \pi_t \in \sigma(r_0, \dots, r_{t-1}), \quad t = 0, \dots, T-1$$

$w_0 \geq 0$ - initial wealth, $u_{1,2}$ - utility functions,
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$u_2 \equiv 0$, $T = \infty$, u_1 is of a special type (CARA, CRRA), $r_{t,0}$ is risk free, $r_0, \dots, r_T$ independent not a.s. above $r_{t,0}$ ...

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2. Directly, using non-anticipativity constraints (possible only for discrete distributions).

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[Merton, 70's - Nobel Prize 1997]

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$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{N} \sigma_{i,j} dW_i(t)\right], \quad i = 1, \ldots, N,$$

$W(t)$ is the N-dimensional Wiener process

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The Optimization Problem

$$\sup_{c,\pi} \mathbb{E}\left[\int_0^T u_1(c(t),t)dt + u_2(w(T))\right]$$

$$c \geq 0, \pi \in \mathbb{R}^N \quad \text{- adapted well behaved}$$

$$dw(t) = \pi(t)dS(t) + [(w(t) - \mathbf{1}'\pi(t))r(t) - c(t)]dt$$

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Theorem

Under some regularity conditions, the solution is closed form and the proportions of risky assets in $\pi(t)$ do not depend on $u_{1,2}$.

Drawbacks of the stock price models

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  $\Rightarrow$ *attempts to incorporate the properties into the models*

Structural issues

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  $\Rightarrow$ *e.g. my present work*

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Radical solution: completely random behavior of the agents (Farmer - still waiting for his Nobel Prize. . .)

Related reading

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